Differential Games

A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization

Rufus Isaacs

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To my wife, Rose,
whose patience with the travail and dislocations
occasioned by this work was indispensable
for its completion

Foreword

The halfway mark in the twentieth century was the beginning of the halcyon period of game theory. With the pioneering work of von Neumann and Morgenstern, *The Theory of Games and Economic Behaviour*, this new mathematical tool became the central focus of the research efforts of the major military analysis groups. Under the leadership of the Mathematics Department of the Rand Corporation, the theory was developed, expanded, and applied to military problems.

Midcentury was also the time of inception of the guided interceptor missile and concentration on the general problem of pursuit and evasion. Is the best way to reach a moving target collision course or constant bearing navigation, direct or anticipatory pursuit? What is the best strategy for evasion? How can two airliners best cooperate to avert collision?

From the forming and solving of military pursuit games in the early 1950's, the present theory grew. Reflecting on the general problem of pursuit, Dr. Isaacs realized that no one guidance scheme could be optimal against all types of evasion, for the evader can deliberately maneuver to confound the pursuer's predictions. Optimal pursuit and evasion, then, must be considered collaterally and with parity; neither could have meaning without the other. Game theory was an essential element of the problem.

If either craft is controlled in flight, by either a human or automatic pilot, there must be certain quantities continually under his volition. Such quantities are unpredictable to the opponent; a guidance scheme based on them is not only futile but against a clever foe may be harmful.

There followed the vital distinction between state and control variables, which was a key step to a general formulation. At once the nature of a strategy was clear: one makes the control variables functions of the state variables. Not only is such an immediate generalization of the strategies of discrete games, but it is also exactly a guidance scheme.

It soon became evident that the resulting formulation was general enough for a wider range of material and Isaacs sought diverse conflict problems and adapted them to his new structure. It was at this time, roughly 1951, that the author coined the name *differential games*.

From the outset Isaacs emphasizes the importance of obtaining answers. In *Differential Games, A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*, not only does he develop the theory of differential games, but his work is enhanced by a magnificent range of practical military and other applications with their solutions. This work is the culmination of fifteen years of reseach by the author and is likely to stand for many years as the definitive work on the subject.

FRANK E. BOTHWELL
Chief Scientist
Center for Naval Analyses
The Franklin Institute

Washington, D.C.
October, 1964

Preface

Although combat problems were its original motive, this book has turned out to be far from a manual of military techniques. Rather the result is a mathematical entity which fuses game theory, the calculus of variations, and control theory, and, because of its subsuming character, often transcends all three.

As ideas burgeoned, they dictated their own course. Combat problems treated as genuine two-player games can be extremely difficult. (This matter is discussed at length in Chapter 11.) Their resolution demanded first a theory and it, one may logically say, is this book's real contribution.

But it grew from solving problems. Each seemed accompanied by strange, new phenomena and, as each conception was mastered, still newer ones proliferated. The unanticipated was and still is a fascinating aspect of differential games. Baffling novelties never seem to cease appearing, and it is therefore yet hard to say how complete the theory is.

To the reader seeking an introduction or a superficial acquaintance with the subject, I suggest the following program:

Chapter 1 depicts the nature and scope of differential games. It presents typical problems, but says nothing about the mathematics or techniques needed for their solution.

The latter can be sensed from Chapter 3, which is devoted to discrete models, some of them being quantized versions of problems later treated continuously. Because these discrete problems can be solved step by step, the reader can glimpse the concepts here without most of the formal mathematical tools.

Chapter 2 essentially casts material such as appears in Chapter 1 in a formal mathematical mold, but the real theory does not begin until

Chapter 4. Thus 1, 3, and a sketchy reading of 2 might be a good chapter order for casual acquaintance.

The reader interested in military applications can turn to Chapter 11, either at once or after the foregoing prelude. The questions of what can be, should be, and may be done to attain military utility fill the early pages. The later ones, which contain specific illustrations, require the technology of the text for their understanding; the reader can stop when he gets there.

Possibly the lack of existence and uniqueness theorems[1] will seem a heresy to some. The emphasis on the specific problem, although counter to current mathematical trends, I feel is good and, in this case, fitting. Without it, it is hard to see how the innovations of the theory could have come to light. Besides, the very opulence of their diversity would seem to preclude the above types of theorems, for such would run to unwieldy lengths were they to cover all cases. The applications themselves have become more diverse than I had at first dreamed, as the reader can discover by skimming the pages.

Between its first publication—the Rand Reports [1],[2] revised versions of which now constitute Chapters 1, 2, 3, and 4—there was a lapse of several years when I had no opportunity of giving differential games the concerted attention it required.

An exceptional resumption occurred when I was on the staff of the Hughes Aircraft Company. The theory of collision avoidance between aircraft and ships is much more recondite than the unitiated might suspect. An investigation, spurred by a series of headlined catastrophes, revealed an unexcepted and elegant liaison with differential games. With cooperation rather than conflict between the two players, collision avoidance problems cohere to the same mathematical principals as games, providing "maximax" replaces minimax. Lack of space precludes the subject from the ensuing text; a separate publication will follow.

As far as I was aware when writing it, this work was an original conception. But unavoidable delays in publication have possibly dimmed some of the sheen of its novelty. As has happened often before in the history of science, at the proper era the same concepts arise simultaneously and independently from widespread investigators. This work was largely a solitary task, and I was unaware of contemporary developments by others.

In fact, it was just a few days after the completion of this manuscript (in March 1963) that I first saw the book [2] by Pontryagin and others,

[1] The alternative used I have called the verification theorem. Its statement, proof, and a demonstration of its use are in Chapter 4.

[2] Bracketed numbers refer to reference list at the end of the book.

which deals with minimizing problems through the same basic devices as presented herein. The technique could be classed primarily as that of one-player differential games.

In his dissertation [3] of 1961, D. L. Kelendzeridze extended this technique to two players, and so to some extent his work tallies with mine.

Besides these Soviet authors, the American contributions [4] deal largely with the logical foundations of the subject. Berkovitz applies the calculus of variations to the strategy of one player, the opponent's being temporarily fixed. Fleming conceives a continuous strategy as lying between two discrete ones. These interesting devices of mathematical rigor appeared too late for their due incorporation in the present work.

However, another native theory, roughly contemporaneous with this one, sounds so distinct that, while writing this book, I did not suspect an affinity. As developed by LaSalle and others [5], control theory is tantamount to that of one-player differential games and thus is largely a special case of the latter. I altered two terms of my Rand Reports to the present *state* and *control variables* in accord with control usage. The *switching surfaces* of this theory are similar to the *singular surfaces* of differential games. The question of *controllability*—what states can be reached from a given starting one—is essentially a specialization of *differential games of kind* (Chapters 8 and 9). Thus each science may enrich the other: control problems can be extended to games by adjoining an opposing player; the ideas of this book are applicable to control material by suppressing one player.

However, I have retained, from the early, genetic problems which suggested them, the names *Pursuer* and *Evader* for the two players, whatever be the nature of the game. An ensuing drawback has been the occasional impression of readers that the subject entails pursuit games exclusively.

Without the allocation of time for the task by my then employers, the Institute for Defense Analyses, this volume would not have been completed. A vital part of my huge debt to this organization is owed in particular to Professor Bernard Koopman, now Director of Research, for his recognition of whatever value this work may have, both as a military tool and a mathematical theory. His willingness to accept a new idea, despite its unconventionality, is a virtue essential to our nation today.

To my present employer, the Center for Naval Analyses, my gratitude is due for first bringing the work to published status as *Differential Games, Research Contribution No. 1*, on 3 December 1963 and for their tolerance in granting the time for final revisions.

My thanks are due to Haig Kafafian of the Center for Naval Analyses for his beneficial advice. To Professor Clifford Marshall, who, acting as referee for SIAM, offered valuable suggestions with as much understanding as an author could wish, and to Professors Harry Hochstadt and R. F. Drenick of this organization, for their enthusiastic cooperation, my deep appreciation. Also, my thanks to the many typists who coped with my symbolism, especially to those such as Mrs. Katherine Gibbon who mastered it.

RUFUS ISAACS

Washington D.C.
October, 1964

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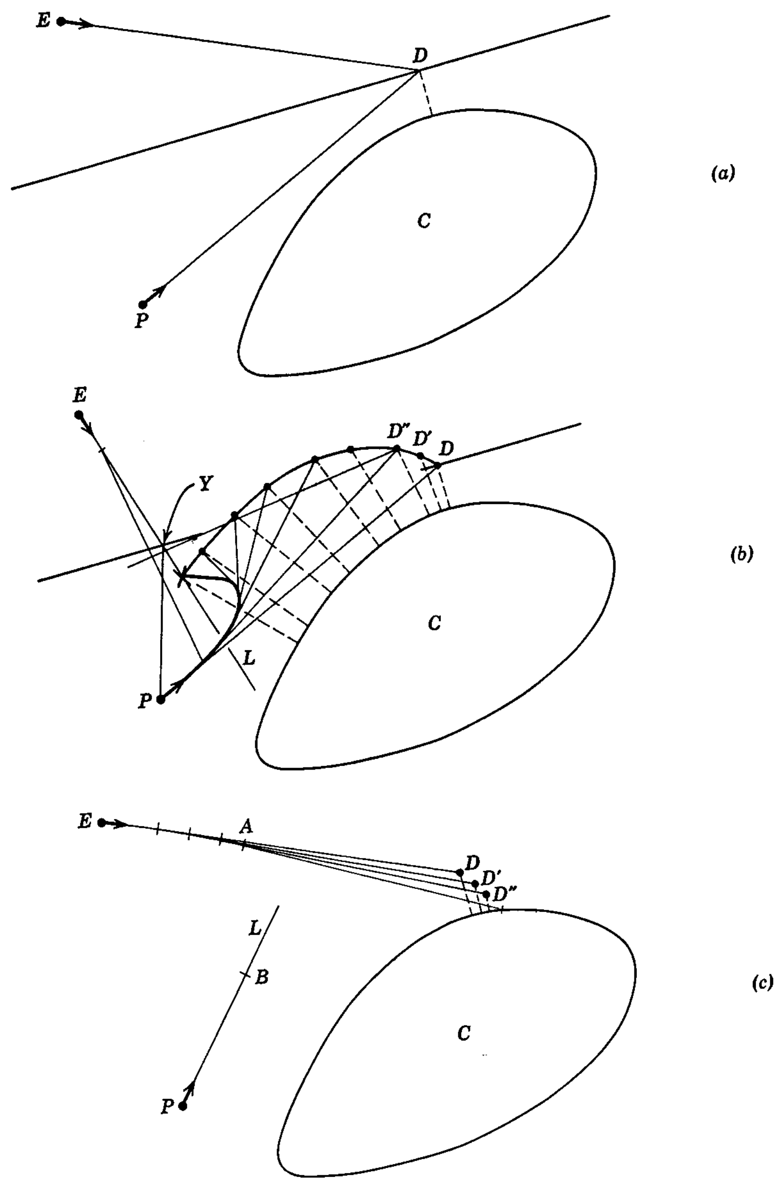


Figure 1.9.2

mathematical model at no time gives P the right to assume that E will persevere on L.

Suppose E remains oblivious of the predatory P until 2 time units have elapsed. Alerted then, be belatedly assumes his optimal strategy, leaving L and fleeing directly away from P. These alternatives are shown on (c) and (d), capture occurring at ×′ with payoffs 9.3 and 10, respectively. The former figure is the best that P can attain if he exploits all the knowledge at his disposal and no more.

The same situation prevails throughout the play depicted at (c). At each instant P acts as if he will face optimal opposition in the future. This trait of an optimal strategy applies to both players and in all pursuit games as we shall formulate them.

If E is known to be unproficient at detecting pursuers and we wish to take this into account, the thing to do is formulate a new game. We might, say, estimate a probability distribution for the time when E is alerted to P's presence. In the new game the payoff would be the expected value of the capture time and P, presumably, will have a new optimal strategy.

On the other hand, when P notes E's oblivious course along L, is he to construe it to mean that E has no detection or evasive capacity at all? If so, of course, the collision course strategy of (d) is the best. Again a revised game might be constructed incorporating an estimated probability that E is impervious to P's presence. But such seems a jejune approach; this is simply a case where a human pilot's judgment may excel a formal strategy.

Example 1.9.2.[15] **Guarding a target.** Both P and E travel with the same speed with simple motion. The motive of P is to guard a target C, which we take as an area in the plane, from an attack by E. The payoff is the distance from C to the point where capture occurs. We take matters to be as in Figure 1.9.2a, where P and E denote starting positions.

The optimal strategies are found thus: Draw the perpendicular bisector of PE. Any point in the half-plane above this line can be reached by E prior to P, and this property fails in the lower half-plane. Clearly E should head for the best of his accessible points. Let D be the point of the bisector nearest C. The optimal strategies for both players decree that they travel toward D. Capture occurs there, and the length of the dotted line is the Value of the game.

Let us see what happens if P plays optimally, but E does not; say he decides to traverse the line L of Figure 1.9.2b. Always P heads for the nearest point to C on the perpendicular bisector, now drawn relative to the

[15] This is the problem posed on page 10.

current positions of P and E. Some typical positions of this point are marked D, D', D'', ... Note the increase in the lengths of the dotted segments. They represent the progressive penalty E pays for his poor strategy. Each length is the payoff were E at the corresponding instant to revert to optimal play and hence is the best that E can hope to recoup.

As P aims at the moving point D, (D', D'', ...) he describes a curved path[16] until D reaches L. From then on D remains fixed, both players move straight, and, in fact, the play is optimal on both sides. Capture occurs at ×.

The same discussion of unwarranted prognosis, given for Example 1.9.1, applies here. If P had advance assurance that E would never leave L, his best policy would be to travel straight to Y (where the bisector meets L). As it is, he swerves to the right before capture. He knows that when E is near his starting position, the upper right part of C is most vulnerable and he moves to protect it; as E advances, the danger relents and P moves in for the capture.

At (c) we see P playing nonoptimally, traversing a line L. Now E should always head toward the moving point D. The dotted segments shrink; when E is at A, D actually reaches C. From then on, E moves straight toward this point. He actually reaches C; nothing P can do now will stop him. (P is at B when E is at A.)

1.10. A PERSPECTIVE ON PRECISION

Although it may be a shock to some mathematicians, to others perhaps a heresy, this work contains no existence theorems. In fact, theorems and lemmas themselves appear on these pages with a much lower density than is common. A sketch of the history of the subject's growth explains why.

From the outset we were interested in delineating and working with as large a class of problems as we could that would satisfy two requirements: they would have some kinship with reality and they could be solved. Thus our goal would be the obtaining of answers rather than the framing of theorems.

The sought kinship did not always turn out to be severely utilitarian. The ideas themselves, as they were gradually revealed, tended to dictate the course of further researches. The homicidal chauffeur game is typical. We can hardly call it a problem in applied mathematics and pinpoint its particular application. Yet when we wish to explore the domain of pursuit and evasion under various type kinematic constraints, bounded radius of curvature seems inevitable. To start, we so restricted one craft; hence the foregoing game. It points the way to the problem of two craft with curvature limitations, the "game of two cars" (Section 9.2), which, although the innovations of principle are manageable, entails calculations difficult to perform by elementary means. Yet this problem is just what is needed in certain applications, such as that of collision avoidance between moving craft, which lack of space forced us to omit from this volume.

[16] The reader is invited to find this path for himself by means of a stepwise, geometric, approximate construction.

But let us see what happened when such problems were attacked with their answers as quarry. The standard scheme of differential equations, as will be expounded in Chapter 4, was the first significant result to emerge. There were a number of intermediate stages, of course, where we were often guided by discrete models, such as those of Chapter 3.

But it rapidly became forcefully evident that in many cases the differential equations alone hardly sufficed. Various kinds of special or singular behavior were often of dominating importance. In the playing space $\mathscr{E}$, these special phenomena most commonly occurred on surfaces. A "surface" here, and in the future, means an $(n - 1)$-dimensional manifold in a n-dimensional space. Besides these singular surfaces, there can also exist such manifolds of lower dimension, but we have largely neglected the latter both because there had to be some limits to our efforts and the former seemed the more interesting. This because they are capable and usually do partition $\mathscr{E}$ into separate regions.

The sorts of singular surfaces that come to light proliferated and each species seemed to have its own distinct theory. A classification scheme will be explained in Chapter 6, but it merely catalogues possible types rather than develope a synoptic theory.

First came barriers (Chapter 8) and universal surfaces (Chapter 7) and then others. Their special yet diverse theories will occupy many of the later chapters. But each first arose in a particular problem; such was always the seed of more general ideas. There were several occasions when we thought the types, for practical purposes, exhausted. But subsequent problems incurred further novelty. At present we do not know how much terrain there is yet to be explored.

Thus the general view of the typical solution of a differential game seems first that the playing space is cut by a number of diverse singular surfaces which subdivide it into a number of component regions. Within each component, the solution may not exist at all, but if it does, it satisfies certain differential equations with boundary conditions picked up at the bordering singular surfaces. The optimal paths—such is the route of the descriptive point $\mathbf{x}$ in the space when both players act optimally—when they exist with a reasonable claim to uniqueness, may have sharp corners when (and if) they encounter a singular surface. Besides all this, it may be that some component regions contain singular manifolds of smaller dimension than surfaces, and indeed such manifolds can be imbedded in the singular surfaces themselves.

With this disparate congeries of singular behavior, it is hard to see how there could be an existence theorem that would embrace all possibilities. Any that was adequate in principle would probably require such a colossally lengthy hypothesis as to render it pedantic in practice.

Thus we have dispensed with attempts to frame such a theorem and substituted another idea which we feel is better suited to the ends of this subject.

These pages will contain a fairly unified technique for getting what, for the moment, let us call formal solutions. A large number of examples will exhibit the process in action. The problem before us thus becomes that of proving that for the formal solutions are, in some reasonable sense, actual solutions.

Two items are required—a satisfactory precise definition of a solution in this "reasonable sense" and a technique for showing that formal solutions conform to it. The first is accomplished through the concept of a K-strategy to be defined in Chapter 2, which also contains what is termed the verification theorem. Judicial use of the latter, often several times in the same problem, can be used to show that for any particular example for which we have found the formal solution, the latter can be converted into the essential constituent of a meaningful, K-strategic solution.

This approach, although unorthodox, seems an appropriate one for the present subject. Of course, we cannot and do not make any general claims as to the the universal existence of solutions of differential games. In an earlier draft, the opening chapters contained several examples displaying various types of nonexistence, but these pathological examples have since been expunged in favor of the more positive cases that can actually be solved. And this task is our real goal.

1.11. A PERSPECTIVE ON PROGRESS

The foregoing goal, actually seeking the solutions of particular examples, has proved rewarding. Phenomena were encountered that, as far as we know, were unprecedented. Sometimes these were extraordinarily baffling, even the very nature of the sought solution being an enigma. We mention a few explicit instances.

The "swerve" maneuver in the homicidal chauffeur game possesses convincing heuristic evidence of its existence; yet what of its quantitative features? How much should the pursuer first turn and how far go straight? In what direction should the evader follow and for how long? The

resolution of these questions came only with the discovery of what we call equivocal surfaces (see Chapter 10) and nothing like them can exist in one-player games or the calculus of variations.

For a long time the most baffling problem of all occurred in the isotropic rocket game, which first makes its appearance in Example 5.3. It differs from the homicidal chauffeur game only in that the pursuer navigates now by controlling the direction of a driving thrust of fixed magnitude, but this difference has very marked effects. The hard question was the game of kind—the conditions under which the pursuer could always capture the evader as distinct from those where the latter could always escape. It seemed intuitively undeniable that the second possibility would hold if the fixed parameters (thrust magnitude, evader's speed, etc.) sufficiently favored the evader. As in the homicidal chauffeur game, E could sidestep whenever threatened by a faster but less agile pursuer.

Now the playing space $\mathscr{E}$ can here be reduced to three dimensions. The set of points in it for which such sidestepping is possible appeared to be externally bounded by a surface resembling a semi-infinite, tapering tent (for details and drawings, see Chapter 9). There was an interpretation of this tent in terms of sidestepping so natural that it was hard to doubt its correctness.

Yet it was open at one end! It did not separate the space. One of two things must be true. Either there were paths which connected the two sides of the tent through the open end, and such implied an outré strategy either for the pursuer to circumvent the sidestepping or for the evader to escape when he was in front of a faster pursuer a miniscule distance from its nose, or there was a way of sealing the open end of tent. The first alternative seeming implausible, we made attempt after attempt at the second.

Only a special class of surfaces are eligible to be seals. It seemed at first impossible to pass one through the boundaries of the tent opening. Finally, an altogether different problem—the deadline game of Example 9.5.2—suggested the answer. It is what we have called an envelope barrier (Section 8.5III). It embodies the remarkable feature of intransient vulnerability, for it is comprised of paths where the evader sidesteps and then, instead of leaping away to safety, he must remain at the very borderline of capture throughout a positive interval of time!

With the unfolding of such ideas, we had more tools and the gamut of solvable problems grew. But what are the limits? Rather late, and unexpectedly, still another of these seemingly baffling novelties appeared. Superficially the problem seemed of childish simplicity; in fact, it was coined as an elementary example to illustrate another point. It has been dubbed *obstacle tag* and will be found in Chapter 6. We bequeath this innocuous appearing game, which seems to defy all our present methods, as a problem to the reader.

We can list a few more general tasks still to be done.

The criteria of universal surfaces (Chapter 7) have been found only when the dimension of $\mathscr{E}$ does not exceed four. Not only should this be carried further, but the case where $n = 4$ could very likely be improved.

Singular manifolds of smaller dimension than surfaces should be studied.

But, most important of all, the theory should be freed from the requirement of full information. Some of the difficulties and possibilities are discussed in Chapter 12.

1.12. ON READING THIS BOOK

We have stressed the value of examples as illuminants of broader ideas. There will be many in the ensuing text, which will result in pages filled with formal calculations.

There will naturally be a tendency for the reader to gloss over these rapidly and to get on to new concepts. Even the author has done so when reviewing a part of the manuscript that had been drawer bound for some months. But the examples are too intrinsic a part of these researches for omission. How should they be approached?

First, we have adopted a standard scheme, which will recur often, of writing the basic differential equations which will be explained in Chapter 4. There was a choice to be made here between the economy of eschewing all repetition and the clarity of precise restatement, and we felt it advisable to favor the latter. The labor of writing a few extra lines is not great compared to that required to reconstruct an obscure meaning; actual redundancy results only very seldom. As one acquires facility in treating problems, he can and should make his own abridgements, but they are best avoided in a public exposition.

Second, despite the standardized format, the examples are not routine. Each has significant, unobvious, and nontrivial features of its own, which are apprehended only by plunging into the analysis.

Third, we suggest that certain readers, instead of following the steps of the text, may wish to work out the examples on their own scratch pads. They can use as guideposts the author's experience in having already been over the ground to garner hints from the text. The gain is of both instruction and enjoyment.

CHAPTER 2

Definitions, Formulation, and Assumptions

The concepts entailed in a differential game are translated into the mathematical vernacular and thus made precise. Alternatives are motivated and illustrated by examples. Several assumptions that will prove useful in our later analysis are stated and justified.[1]

Throughout we deal with games of perfect information. That is, at all times each player knows the values of the state variables.

2.1. THE KINEMATIC SITUATION

Our theatre of operations is $\mathscr{E}$, a region in Euclidean n-space and its boundary. This boundary is to consist of pieces of certain surfaces (we mean by surface an $(n-1)$-dimensional manifold[2]). We think of a particular point $\mathbf{x} = \{x_1, \ldots, x_n\}$ to be in motion in $\mathscr{E}$, its path being governed by what we shall call the *kinematic equations*, abbreviated KE:

$$\dot{x}_j = f_j(x_1, \ldots, x_n, \phi_1, \ldots, \phi_\lambda, \psi_1, \ldots, \psi_\kappa), \qquad j = 1, \ldots, n \quad (2.1.1)$$

or more briefly

$$\dot{\mathbf{x}} = f(\mathbf{x}, \phi, \psi). \quad (2.1.1)$$

The functions f are given; we suppose them to be of a simple character, and in the sequel we shall not hesitate to speak of any partial derivative (of any order) of the f_j that we have occasion to need. We term ϕ and ψ the *control variables*. They are at all times one each under the control of a

[1] Originally Rand Report RM-1399 (30 November 1954). Certain additions and improvements have been made and several pathological examples deleted.

[2] Supposed piecewise smooth.

player. Thus the motion of **x** is to be thought of as influenced by the wills of two individuals. If they seek conflicting objectives—and only such cases are of interest[3]—the situation assumes something of the nature of a game. As suggested by game theory, we will speak of a particular **x** in $\mathscr{E}$ as a position or state; we call the $x_1, \ldots, x_n$ the *state variables* in that they describe this state; the two individuals are the *players*.

The x_j are descriptive in the following sense. If a play of a differential game is halted before completion, the values of $x_1, \ldots, x_n$ at the time of interruption supply all the data needed to resume the partie. We mean that if a new partie is commenced starting with these x_j, it will be tantamount to the part of the original that would have occurred after the interruption.

In particular, the values of x_j at the outset supply starting data. Thus when we use the term *game*, we are not speaking of a single game but of a collection. There is a distinct game emanating from each point of $\mathscr{E}$.

In general, the ϕ and ψ are (individually) subject to certain constraints which depend at most on **x**, the typical form being: $a_i(\mathbf{x}) \leqslant \phi_i \leqslant b_i(\mathbf{x})$. It will always be understood that we are interested only in ϕ and ψ which satisfy the constraints, and explicit mention of them in the future will be considered unnecessary.

With ψ fixed and **x** fixed in $\mathscr{E}$, the set of vectors $f_j(\mathbf{x}, \phi, \psi)$ for all ϕ will be called a *vectogram* or a *ϕ-vectogram* (similarly for a *ψ-vectogram*). A *full vectogram* allows both ϕ and ψ to take all values.

For example, planar simple motion is depicted by a circular vectogram of fixed radius (= the speed) at each point. Such a ϕ-vectogram[4] is shown at Figure 2.1.1a. At (b) is shown a simple prototype of a full vectogram with $n = 3$ and ϕ, ψ having one component each.

Presupposing the numerical payoffs to be introduced in Section 2.4, we name the players:

P, controlling ϕ and minimizing.

E, controlling ψ and maximizing.

The names relate to the Pursuer and Evader of pursuit games. Although such games were the seeds of the present theory, its scope now embraces a wide range of phenomena, as the subsequent examples will show. Because we have retained the designations above, let not the cursory reader infer that contests of pursuit and evasion are synonomous with differential games. But pursuit games, especially in those early chapters, are fine vehicles for illustrating points of the theory in general.

[3] There can be exceptions, such as in the theory of collision avoidance where the participants cooperate.

[4] When a control variable has but one component, we shall omit the subscript.

The names, *P* and *E* for the players in general types of problems have a less sterile ring than such nomenclatures as Player I and Player II, or Red and Blue. Our choice gives the players individualities—personalities, if one prefers—without sacrificing a certain symmetry of roles which is, in a sense, the essence of game theory, as opposed to one-player problems.

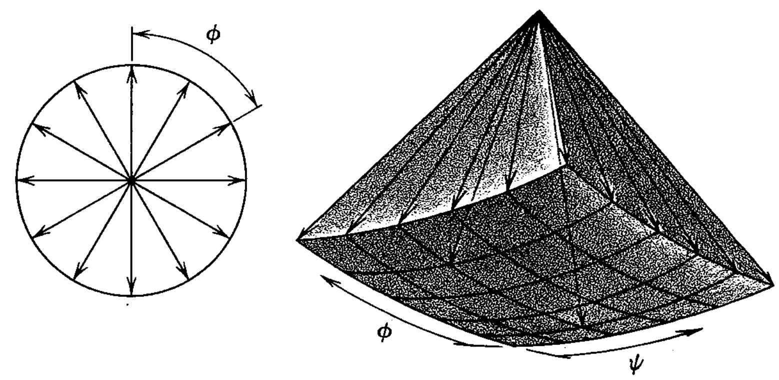
The obvious mnemonic as to roles is in terms of pursuit games with time of capture as payoff: *P* minimizes, *E* maximizes.

When working problems we shall often use more descriptive letters than $x_1, x_2, \ldots$ for the state variables. Standard coordinates such as (x, y) or (r, θ), for example, can designate a point in the plane and subscripts may be added if there are several points. Or we will use initial or other suggestive letters to denote entities, such as men, munitions, and time.

Example 2.1.1. Planar pursuit games with simple motion. In both Examples 1.9.1 and 1.9.2, if we let (x_1, y_1) be the coordinates of *P* and as (x_2, y_2) those of *E*, and w_1, w_2 their respective speeds, the KE can be written

$$\dot{x}_1 = w_1 \sin \phi$$
$$\dot{y}_1 = w_1 \cos \phi$$
$$\dot{x}_2 = w_2 \sin \psi$$
$$\dot{y}_2 = w_2 \cos \psi.$$

Example 2.1.2. The homicidal chauffeur game. A natural formulation of the problem requires five state variables: two coordinates each to specify the position of *P* and *E* and one more to specify *P*'s flight direction. They



(a) (b)

Figure 2.1.1

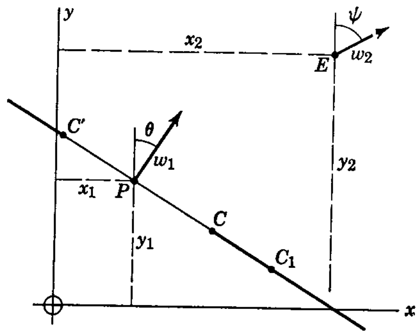


Figure 2.1.2

can be taken as $x_1, y_1, x_2, y_2, \theta$ as shown in Figure 2.1.2. Such specify completely and uniquely determine the position at any instant of play. Turning to the control variables, that of E is the simpler. One, ψ, is needed to describe his flight direction as shown. For that of P, draw the line $C'PC$ perpendicular to P's velocity vector. The volition of P allows him to choose his current center of curvature at any point, such as C_1, on this line with the open segment $C'C$ excluded ($|C'P| = |PC| = R$). Typically, we define his one control variable ϕ as $R/|PC_1|$ so that $-1 \leqslant \phi \leqslant 1$.

Then the KE (kinematic equations) are

$$\begin{aligned}
\dot{x}_1 &= w_1 \sin \theta \\
\dot{y}_1 &= w_1 \cos \theta \\
\dot{x}_2 &= w_2 \sin \psi \\
\dot{y}_2 &= w_2 \cos \psi \\
\dot{\theta} &= \frac{w_1}{R} \phi.
\end{aligned}$$

A player may have more or less control of his present and future, but no one can affect the past. Thus we interpret the left sides of each KE as a forward time derivative.

2.2. THE REALISTIC AND REDUCED SPACE

When we construct a model from a physical prototype, the set of state variables will generally be such as to convey a direct and forthright description of the situation. Their number n, the dimension of $\mathscr{E}$, however, may be larger than necessary. Often by judiciously picking a less direct set of coordinates n may be lowered. When such has been done, we will speak of the *reduced space*. We will use the symbol $\mathscr{E}$ for it as well as for

the original, which we shall call the *realistic space*.[5] In either case, in general discussion, n will be the dimension.

The advantages of a reduced space, with its smaller number of state variables are less redundancy and required writing. Also, if n can be made 3 or less, the convenience of geometric visualization is sometimes a great aid in intricate situations. But there are arguments in favor of the realistic space.

The KE, although more numerous, are sometimes much simpler. If the problem involves moving craft, their paths in the realistic space are what they are physically; in a reduced $\mathscr{E}$, a very simple such motion, such as in a straight line at constant speed, may appear almost unrecognizably complex.

Example 2.2.1. If, in Example 1.9.2, the target area is a half-plane, say that below the y-axis, we can employ a reduced space with $n = 3$ instead of 4 as in Example 2.1.1.

If we put $x = x_1 - x_2$ it is clear that knowledge of x, y_1, y_2 suffices to describe a state. The KE then become (In Example 1.9.2 the speeds were equal; if such is desired here put $w_1 = w_2$):

$$\begin{aligned}
\dot{x} &= w_1 \sin \phi - w_2 \sin \psi \\
\dot{y}_1 &= w_1 \cos \phi \\
\dot{y}_2 &= w_2 \cos \psi.
\end{aligned}$$

Exercise 2.2.1. In the preceding example show that if the target area is circular, say centered at the origin and of radius R, then n can similarly be reduced to 3 and write the corresponding KE.

Exercise 2.2.2. Show that in Example 1.9.1 there is a reduced space with $n = 1$.

Observe that such reductions may not be possible if the realistic space should lose its homogeneity, say if the speeds were functions of x, y.

Example 2.2.2. The homicidal chauffeur game. A reduced $\mathscr{E}$ can lower n from 5 to 2. Let us think of a map of the realistic space (the "parking lot") as affixed to the car P. We can use the coordinates x, y of E's position, which will be $\mathbf{x}$ on this map, as the sole state variables. The y-axis is always to be in the direction of P's velocity vector.

[5] We prefer to leave these definitions somewhat flexible rather than tie ourselves down by stringent requirements. The usages in subsequent examples will render the idea clear. We have not coped with the general problem of ascertaining the minimal possible n.

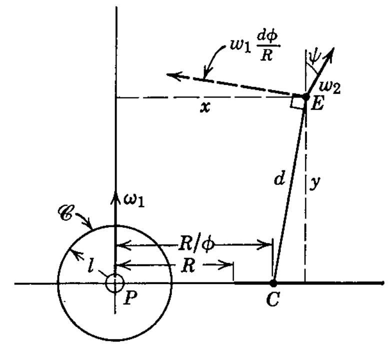


Figure 2.2.1

As in Figure 2.2.1, let P choose his center of curvature at $C = (R/\phi, 0)$ and let d be the distance CE. Then P's rotation about C is tantamount to a rotation of $\mathbf{x}$ about C in the opposite direction with the same angular speed. Thus $\mathbf{x}$ moves with speed $w_1\,(d\phi/R)$ in a direction perpendicular to CE. Its velocity components are obtained by multiplying the speed by $-y/d$ and $(x - R/\phi)/d$. Thus the KE are

$$\dot{x} = -\frac{w_1}{R}\,y\phi + w_2 \sin \psi$$ [6]

$$\dot{y} = \frac{w_1}{R}\,x\phi - w_1 + w_2 \cos \psi, \qquad -1 \leqslant \phi \leqslant 1.$$

We take $\mathscr{C}$ as the origin centered circle of radius l and $\mathscr{E}$ as the portion of the x, y-plane exterior to $\mathscr{C}$. (See the next section.) Note that if E travels straight but P deviously, ψ will be an involved function of the time. Such is typical of the disadvantage of a reduced space.

2.3. TERMINATION OF THE GAME

There is a surface $\mathscr{C}$, called the *terminal surface*, which is part of the boundary of $\mathscr{E}$. When $\mathbf{x}$ reaches $\mathscr{C}$, the game is over.

We take this form of termination as part of the canonical definition but feel obliged to defend our motivation. Why a surface? Why part of the boundary of $\mathscr{E}$?

The termination of a pursuit game usually is capture, which at first glance we might take to mean the coincidence of P, with E.[7] If the x_i are the totality of variables descriptive of the positions of both P and E, the subset of $\mathscr{E}$ corresponding to capture would be of dimension $<n - 1$. We reject this definition of capture on two grounds.

First, it is unrealistic. In applications, the point P or E will be some fixed spot on a missile (aircraft, ship, torpedo, etc.) of tangible bulk, intended to serve as an index of the missile's location. On these grounds alone, P and E will never coincide; but in tactical situations often all that is required for capture is less even than physical contact—just a certain proximity. Thus a more reasonable criterion of capture is, say, to specify a positive l and capture occurs when the distance P to E is l.[8] The set of all capture positions is specified by one equation and thus constitutes a surface in $\mathscr{E}$.

The second ground applies generally. The technique we shall use entails differential equations. A surface provides just the number of dimensions needed for initial conditions so as to obtain unique solutions. A lesser dimensional manifold generally gives rise to singular points of the solutions. (In Chapter 6 are some examples of games which are pathological because of this dimensional deficiency.) Should we be presented with a game with a terminal set $\mathscr{C}'$ of too small a dimension, we amend it by using the boundary of a δ-neighborhood of $\mathscr{C}'$ for a terminal surface. If desirable, we can investigate the limiting situation as $\delta \to 0$.[9]

Suppose that, in the formulation of a game fresh from the physical situation, the surface $\mathscr{C}$ is not on the boundary of $\mathscr{E}$ but is interior to $\mathscr{E}$. Locally it separates $\mathscr{E}$ or, relatedly, $\mathscr{C}$ has two "sides." Often we will wish to count as termination only the cases where $\mathbf{x}$ reaches $\mathscr{C}$ from a particular side. For example, let us return to the above pursuit game, and suppose we started from a position where the distance P to $E < l$. Clearly we would not want to consider a subsequent occurrence of $|PE| = l$ as capture. What we do here is exclude all positions with $|PE| < l$ from $\mathscr{E}$. Then $\mathscr{C}$ will be part of the boundary.

However, there may be cases where $\mathscr{C}$ is desirably in the interior of $\mathscr{E}$. We will then distinguish between approaches of $\mathbf{x}$ to the two sides of $\mathscr{C}$. We can think of $\mathscr{E}$ as "slit" along $\mathscr{C}$ and $\mathscr{C}$ itself as two-sheeted. Thus, in a sense, we have restored to $\mathscr{C}$ its role of boundary.

What shall we do if $\mathbf{x}$ never reaches $\mathscr{C}$? A reasonable and quite practical thing is to supply a stop rule. That is, we select some large value T of time and decree that the game is over should T elapse. We can bring this situation into the canonical picture by introducing time as a new state

[6] Not the same as the old ψ, of course.

[7] Simple pursuit games involve the motion of two points, one pursuing the other, as explained in the ensuing paragraph. We shall use P and E for the names of these points as well as for the names of the controlling players.

[8] There can, of course be cases with "capture regions" of shapes other than circular.

[9] When working problems by simple geometric methods, such as the examples in Chapter 1, the coincidence of P and E seems a convenient theoretical criterion of capture. See Sections 6.7 and 6.8.

variable, x_{n+1}. We enlarge the kinematic equations with $\dot{x}_{n+1} = 1$ and take for the new $\mathscr{E}$ the direct product of the old by $[0, T]$. The new $\mathscr{C}$ is the direct product of the old by $[0, T]$ as well as the part of $x_{n+1} = T$ bounding the new $\mathscr{E}$. We only consider plays of the enlarged game which start from an $\mathbf{x}$ with $x_{n+1} = 0$.

As $\mathscr{C}$ is a surface, an $(n - 1)$-dimensional manifold, we can represent it by $n - 1$ parameters. Our standard general way of so doing is

$$x_i = h_i(s_1, \ldots, s_{n-1}) = h_i(s), \qquad i = 1, \ldots, n. \tag{2.3.1}$$

We shall assume these functions to be differentiable. In our problems, at worst $\mathscr{C}$ will be piecewise smooth; we treat each piece with a separate representation (2.3.1).

2.4. THE PAYOFF

The numerical quantity which the players strive to maximize and minimize in games of degree can assume a variety of guises. We prefer to take as the standard form of the payoff

$$\int G(\mathbf{x}, \phi, \psi)\, dt + H(s). \tag{2.4.1}$$

Here the function G is qualified as are the f_i (as to having partial derivatives, etc.). The time integral extends over the path in $\mathscr{E}$ traversed by $\mathbf{x}$ during a partie; its lower limit (we could call it $t = 0$) refers to the starting point in $\mathscr{E}$; its upper limit is the time of termination—when $\mathbf{x}$ reaches $\mathscr{C}$.

The function H is a smooth one defined on $\mathscr{C}$. For any partie, the second term of (2.4.1) is the value of H at the point of termination—where $\mathbf{x}$ meets $\mathscr{C}$ as play ends.

Of course, if both H and G were 0 we would have a vacuous situation; we exclude it. If $H = 0$, we will say the game has an *integral payoff*; if $G = 0$, a *terminal payoff*. Almost all practical examples are of these two types.

As instances, pursuit games with time of capture as the payoff have integral payoffs with $G = 1$. Example 1.9.2 has a terminal payoff: the distance from E to the target when capture occurs.

For certain theoretical purposes games with terminal payoffs are advantageous and we shall then make use of

THEOREM 2.4.1. A game with a payoff of the form (2.4.1), with $G \neq 0$, can be replaced by an equivalent one with terminal payoff.

Proof. Primed symbols refer to the new equivalent game; unprimed

ones to the original. For $\mathscr{E}'$ we take the direct product

$$\mathscr{E} \times L$$

the L meaning the doubly infinite interval $(-\infty, \infty)$; it is to be the range of a new state variable denoted by x_{n+1}. Similarly,

$$\mathscr{C}' = \mathscr{C} \times L$$

and the parametrization of $\mathscr{C}'$ will be (2.3.1) together with

$$x_{n+1} = s_n. \tag{2.4.2}$$

To the old KE we adjoin

$$\dot{x}_{n+1} = G(\mathbf{x}, \phi, \psi). \tag{2.4.3}$$

The payoff will be terminal with

$$H'(s') = H(s) + s_n. \tag{2.4.4}$$

Now let us consider a partie of the new game starting from $\mathbf{x}^0$, with components x_i^0, and terminating at a point s' of $\mathscr{C}'$. As x_{n+1} is not involved in the first n kinematic equations, if we project the path on $\mathscr{E}$, it will correspond to one resulting from a partie of the old game. Conversely, any partie of the old game corresponds to one of the new. For $x_1, \ldots, x_n$ as well as ϕ and ψ, will then be known functions of t; they can be inserted into (2.4.3), which is then integrated with the condition $x_{n+1}(0) = x_{n+1}^0$.

What is the payoff of a partie of the new game? It is given by (2.4.4), where s' is the termination point on $\mathscr{C}'$, consisting of s on $\mathscr{C}$ and s_n. By integrating (2.4.3) from $t = 0$ (at $\mathbf{x}^0$) to its final value (at s') we have, using (2.4.2),

$$s_n - x_{n+1}^0 = \int G(\mathbf{x}, \phi, \psi)\, dt$$

the latter integral extending over the path in $\mathscr{E}'$ or, what is the same thing (x_{n+1} is not in the integrand), its projection on $\mathscr{E}$. Substituting in (2.4.4) we have

$$\text{Payoff} = H(s) + \int G(\mathbf{x}, \phi, \psi)\, dt + x_{n+1}^0.$$

If we confine ourselves to starting points with $x_{n+1}^0 = 0$, then the payoff will be exactly that of the original game.

Note that no essential generality is lost by this confinement. As x_{n+1} appears on the right in no KE, all paths in $\mathscr{E}'$ which differ in starting positions only in x_{n+1}^0 are translations of the same one in the x_{n+1} direction.

Other types of payoffs are subsumable in the form (2.4.1).

Suppose t (time) effectively appears among the arguments of the f_j, of G, or even of H. In the latter case, the payoff is a function of the time of termination as well as the place. Then we adjoin $\dot{x}_{n+1} = 1$ to the kinematic equations, take the new $\mathscr{E}$ and $\mathscr{C}$ as the direct products of the old by the full interval $(-\infty, \infty)$ of x_{n+1} and replace the argument t in f_j, G or H by x_{n+1}. When the revised game has been analysed we discard all starting points except those with $x_{n+1} = 0$.

There are applications with payoffs

$$\int_0^T G(\mathbf{x}, \phi, \psi)\, dt$$

where T is some prescribed positive value of the time ($t = 0$ means the starting time). Such a T is essentially a state variable. We adjoin to the KE

$$\dot{T} = \dot{x}_{n+1} = -1$$

and take the new $\mathscr{E}$ as the direct product of the old by $[0, \infty)$. For $\mathscr{C}$ we take that part of the boundary of the new $\mathscr{E}$ where $x_{n+1} = 0$. We play the revised game with an integral payoff with integrand G. We utilize only starting points with $x_{n+1} =$ the prescribed value of T.

Suppose we are given a function $K(\mathbf{x})$ defined in $\mathscr{E}$, and the payoff is to be the value of $K(\mathbf{x})$ at the end of a prescribed time T. We treat this case similarly to the preceding but use a terminal payoff with $H = K$.

Another type of payoff which can be reduced to the standard form, not always but at least in simple cases, is as follows: Let $K(\mathbf{x})$ be given in $\mathscr{E}$. The payoff will be the minimum of $K(\mathbf{x})$ which occurs during the play. Example: In a pursuit game, how close can the pursuer get to the evader?

Let $\mathscr{E}_1$ be that subset of $\mathscr{E}$ in which E can cause $K(\mathbf{x})$ to increase whatever P may do. That is $\mathscr{E}_1$ is the set of $\mathbf{x}$ for which

$$\max_{\psi} \min_{\phi} \sum_j K_{x_j} f_j(\mathbf{x}, \phi, \psi) > 0.^{10} \tag{2.4.5}$$

Let $\mathscr{C}$ be the boundary of $\mathscr{E}_1$. It is clear that if a minimum of K occurs at all, against optimal opposition from E, it will occur on $\mathscr{C}$.[11] Thus we may reduce matters to a terminal payoff with H being the value of K on $\mathscr{C}$. The reader can easily construct examples in which P can achieve low minima only by causing $\mathbf{x}$ to enter $\mathscr{E}_1$ and leave it again; in such cases the idea just given will present its difficulties.

For a vector $\mathbf{u} = (u_1, \ldots, u_n)$ write for short

$$Q = \sum_i u_i f_i(\mathbf{x}, \phi, \psi) + G(\mathbf{x}, \phi, \psi).$$

[10] Unspecified limits of summations will always be 1 and n.

[11] We are supposing all given functions continuous, differentiable, etc. In general, $\mathscr{E}_1$ will be open and $\mathscr{C}$ a surface. In fact, $\mathscr{C}$ will be defined by (2.4.5) with $>$ replaced by $=$.

Vital for our work is the

MINIMAX ASSUMPTION. For all $\mathbf{u}$ and all $\mathbf{x}$ in $\mathscr{E}$

$$\min_{\phi} \max_{\psi} Q = \max_{\psi} \min_{\phi} Q.$$

In all applications encountered up to the time of writing, each f_j and G have been *separable*, that is, of the form: function independent of ψ + function independent of ϕ. In such cases the minimax assumption obviously holds.

2.5. GAMES OF KIND AND GAMES OF DEGREE

When we speak of a *game of degree*, we mean one with a continuum of possible payoffs such as in the last section. A *game of kind* has finitely many, usually only two, the outcome of the game depending on whether or not one of the players can achieve some objective. For example, in a pursuit game the objective might be capture; in a battle game, complete extermination of the opponent.

If a stop rule is imposed, a game of kind becomes one with terminal payoff for which H assumes only finitely many values. The game falls within our compass and no special treatment is required. Nevertheless, it is often possible and desirable to imbed the game of kind within one of degree and deal with the latter.

The solution to a game of kind may be tremendously indeterminate. It results in the division of $\mathscr{E}$ into two (sometimes more) aliquot subsets (some possibly null), one favorable to each player. If the starting point lies in a player's set, he can attain his objective. Then usually any strategy is optimal for him as long as it lets him remain in his set, whereas any strategy at all is optimal for his opponent. These ideas will be developed at length in Chapter 8.

Let us take two species of games of kind:

1. A pursuit game with capture as the objective.
2. The same game with the objective capture before a stipulated time T. Such would be the case, say, if P were an interceptor missile with a limited fuel supply.

In both cases we would lose nothing and might gain much if the time of capture is made the payoff, which is taken as ∞ if capture does not occur. We can then expect definite optimal strategies instead of a sprawling class delineated only by inequalities. In case 1, the strategy will not only instruct P as to how to capture but will show him how to do it as quickly as possible. Similarly, it tells E how to delay it. If we have case 2, we need only look at the Value[12] and see whether or not it exceeds T. We will have solved case 2 for all possible values of T at once.

However, we do not advocate abandonment of the game of kind; in fact, some later chapters will be devoted to them. There are cases where the direct solution is much simpler than the embedding procedure just suggested and supplementary information of little value. Sometimes, too, a game of kind appears as a phase of a game of degree. For example, a player may not be able to terminate favorably unless he first surmounts some obstacle. The question of whether this can or cannot be done may constitute a game of kind whose solution is a preliminary to that of the whole game.

When we speak of a game without specifying whether it is of kind or degree, it will be understood to be of the latter type.

2.6. STRATEGIES

In the theory of discrete games, a strategy is defined as a set of decisions for a player, one for each position that may arise. If each player chooses a strategy, the ensuing partie, and particularly the payoff, is uniquely determined.

We recognize a somewhat analogous circumstance existing here. The election of a decision for each possible position amounts to a player's choosing his control variables as functions of the state variables. If the players each so select $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ and these values are inserted in the kinematic equations, the latter become differential equations. Recalling that the data of a game must include a starting value of $\mathbf{x}$, we see that this value plays the role of an initial condition. Thus, under reasonable circumstances, we may expect the paths—and hence the payoff—to be uniquely determined.

As in general game theory, the Value of the game is to be the minimax of the payoff. Symbolically

$$V(\mathbf{x}) = \min_{\phi(\mathbf{x})} \max_{\psi(\mathbf{x})} \text{ (payoff)}.$$

Here the min [max] extends over all allowable strategies $\phi(\mathbf{x})$, [$\psi(\mathbf{x})$]. We are to expect that the min max equals the max min, an expectation that will be justified later on. (It depends rather strongly on the minimax assumption of Section 2.4.)

We shall always capitalize the term Value so that we may retain literary usage of the word "value." The Value is a function $V(\mathbf{x})$ of the starting point $\mathbf{x}$, a function that will play a key role in our subsequent analysis.

[12] The Value of the game, a standard term in game theory. It is fitted to our context in the next section.

These concepts carry a measure of heuristic satisfaction; but we must not probe them too much with logical precision.

At each instant during the course of a play, the players are faced with a full vectogram. If we think of each as choosing a value of his control variables, the choice results in the selection of a constituent velocity vector along which $\mathbf{x}$ will travel in the immediate future. Thus what corresponds to the moves in a discrete game is here a continuous and unrelenting choice of ϕ or ψ. The reader may justly protest that we are demanding of the players feats beyond human ability and of the mathematics problems beyond rigorous analysis. We shall assuage him shortly.

An attempt to define strategies in the form $(\phi(\mathbf{x}), \psi(\mathbf{x}))$ leads immediately to difficulties. First we require assurance that the differential equations to which the KE are converted are integrable. We recall that the left sides are to be construed as forward derivatives. Now the existence criteria for "forward differential equations" are much broader than those of the classical theory and their limitations quite distinct, as the following examples will indicate. Let us take first

$$\dot{x}_i(\text{forward derivative}) = f_i(x_1, x_2), \qquad i = 1, 2$$

$$\begin{aligned} \text{where } (f_1, f_2) &= (1, 1) \text{ when } x_1 < 0 \\ &= (0, 2) \text{ when } x_1 = 0 \\ &= (-3, 0) \text{ when } x_1 > 0. \end{aligned}$$

The reader will see that this system has exactly one solution starting from each point in the plane. Later we will find that functions of this genre are no strangers to the solutions of differential games.

Now let us take for f:

$$(1, 1) \text{ when } x_1 \leqslant 0$$

$$(-3, 0) \text{ when } x_1 > 0.$$

We are frustrated should we start from or arrive at a point with $x_1 = 0$.

A theory of such differential equations now claims a number of papers,[13] but we have not followed this possible route in our work.

In the sequel we shall develop methods of solution. The results will include values of ϕ, ψ, say $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$, which we shall term optimal.

[13] H. Bilharz, *Z. angew. Math. Mech.* **22** (1942), 206–215, D. Bushaw, Contributions to the theory of nonlinear oscillations, IV, Princeton 1958; Flügge-Lotz, Discontinuous automatic control, Princeton 1953; J. André and P. Seibert, *Archiv d. Math.* **7**, 148–156, 157–165 (1956), *Comptes Rendus* 245, 625–627 (1957) Boletin de la Sociedad Matematica Mexicana 242–245 (1961). Solncev, Moscow. Gos. Univ. Učenye Zapaki, 48, Mat. **4**, 144–180 (1951). A more detailed exposition of the subject will appear in the "Contributions to the Theory of Nonlinear Oscillations," Vol. V, edited by S. Lefschetz.

When inserted in the kinematic equations, these strategies render them at least piecewise integrable (or integrable in the sense of forward differential equations). Solutions (paths, the payoff, etc.) can generally be computed, and they appear to be optimal in the sense of rendering the payoff a minimax.

But there remains a second difficulty. An assertion that, say, $\bar{\phi}$ is optimal entails knowledge of its performance when opposed by a certain class of ψ. What class? It must be such that the pair $\bar{\phi}$, ψ will always lead to integrable KE and that all ψ representative of an actual player's actions are included.

Samuel Karlin has advanced an idea which obviates these troubles. A strategy for P is defined as both the choice of a function $\phi(\mathbf{x})$, now subject to no restrictions (save the constraints) and a sequence $\sigma_t = \{t_0 = 0, t_1, t_2, \ldots\}$ of increasing values of time with $\lim_j t_j = \infty$. Such will be called a *K*-strategy. In playing it, suppose P at time t_k finds $\mathbf{x}$ to be at $\mathbf{x}^{(k)}$ ($\mathbf{x}^{(0)}$ is the starting position). Then in the interlude $[t_k, t_{k+1})$, P holds ϕ to the constant value $\phi(\mathbf{x}^{(k)})$.

Let a *K*-strategy, $\psi(x)$ and $\sigma_t' = \{t_0' = 0, t_1', \ldots\}$ be also defined for E. We subdivide time by both the t_j and the t_j'. In each subinterval both ϕ and ψ are constant and so the KE are obviously integrable. We build the path, using as the initial values for each later interval the final $\mathbf{x}$ from the previous.

Thus for each starting point and pair of *K*-strategies, the path of $\mathbf{x}$, and consequently the payoff, is uniquely determined. We define the Value as the sup inf (= inf sup) of the payoff, the sup and inf ranging over the respective players' classes of *K*-strategies. Such is the natural counterpart of the more primitive minimax definition given earlier in this section.

It is hard to see, in the world of actuality, how a sequence of decisions could be anything but discrete. Thus the *K*-strategies seem a step nearer to reality.

We will speak of the $\phi(x)[\psi(x)]$ which is a constituent of *K*-strategy as a *tactic*.[14]

It is manifest that, in general, the *K*-strategies will not yield optimal strategies but only ε-optimal strategies, that is, strategies that will attain within ε of the Value (this being done, it would seem, by increasing the fineness of the temporal subdivision).

[14] The terms *strategy* and *tactic* have nothing to do with the military lexicon. The former, since its introduction by von Neumann and Morgenstern, has become standard in game theory and we use an obvious extension. We introduce the latter when speaking of *K*-strategies because an alternative word is convenient. Its usage in this book will be quite limited.

Can we be sure that strategies subsume all the best modes of playing? Let us, for the moment, waive subtleties and accept strategies in our earlier sense. Suppose a player, say E, follows the dictates of a policy $S_E{}^*$ not a strategy as we have defined it. For example, $S_E{}^*$, may entail a ψ which is a function of $\dot{x}_j$, other higher derivatives of the x_j (E somehow provides for cases where they no not exist), past values of the x_j, integrals over such past values, etc. If E pits $S_E{}^*$ against an optimal strategy S_P for P, how do we know he will not emerge with better than the Value?

We will endeavor to reply in two ways. The first is heuristic. It is based on the fact that the state variables are truly descriptive of the state in the sense discussed in Section 2.1.

To illustrate, let us consider in part a pursuit game in which P is a point moving in a plane. Let x_1, x_2 be his coordinates. First let him have simple motion so that the kinematic equations are in part

$$\dot{x}_1 = w \cos \phi_1$$
$$\dot{x}_2 = w \sin \phi_1.$$

We claim that the only rational policy for E is to base his actions on x_1 and x_2 alone. He might have them depend on, say, $\dot{x}_1$, $\dot{x}_2$ past values of the x_j etc. as well, such as would be the case if he endeavored to extrapolate P's future positions. But P's velocities—according to the way we have framed the problem—are at all times subject to abrupt change without notice. It is impossible for E to rely on any prediction or indeed to derive any constructive knowledge from anything other than x_1 and x_2.

Let us now make P's motion a bit more complicated. We suppose that now he regulates the acceleration ϕ_1, ϕ_2 (subject to some bounds that we will not mention). The KE in part now might be

$$\dot{x}_1 = x_3$$
$$\dot{x}_2 = x_4$$
$$\dot{x}_3 = \phi_1$$
$$\dot{x}_4 = \phi_2.$$

Now P can no longer abruptly switch velocities, and there are sound grounds for E's basing his policy on them. But the velocities are x_3, x_4 and appear among the state variables. However, the same argument as before shows that E could be misled if he based his decisions on, say, P's acceleration.

We can proceed thus, creating a chain of more and more complex types of motion for P and, in fact, can introduce many variants and offshoots along the way. In each case we single out those data of the motion on which it appears E can rationally rely when making his decisions, and in each case they appear among the state variables.

The second reply is mathematical. Suppose P plays S_P and E plays S_E* from some starting point. On the resulting path we will have the ψ arising from S_E* defined as a function of t. We take the liberty of supposing that this function is piecewise continuous. Then there will be a strategy S_E for E which will agree with S_E* whenever a partie results in this same path. Thus E will reap the same yield if he plays S_E or S_E* as long as P adheres to S_P. As S_P is optimal, E cannot do better than the Value.

2.7. CANONIZATION OF THE VECTOGRAMS

In our science, as in other fields of analysis, we are free to make certain transformations on the variables involved. For example, if $\mathscr{C}$ is a smooth surface, we can select the state variables so the $\mathscr{C}$ lies in the surface where $x_1 = 0$. But we wish to speak here of transformations on the control variables which bring the vectograms to certain canonical forms that will later prove convenient.

We can, of course, and should assume there are no redundancies: on a ϕ-vectogram, say, we choose coordinates (the $\phi_1, \ldots, \phi_\lambda$) so that to each velocity in the vectogram there corresponds but one set of numerical values. It follows, then, that $\lambda \leqslant n$.

Another rather obvious requirement on meaningful vectograms is that they permit sufficient scope to the navigation they allow $\mathbf{x}$ in $\mathscr{E}$, rather than confine $\mathbf{x}$ to a lower dimensional subset. In this event we could reformulate the problem taking for $\mathscr{E}$ this subset.

Example 2.7.1. Let $n = 3$, $\lambda = 1$, $\kappa = 0$. The KE are

$$\mathbf{x} = \alpha(\mathbf{x})\phi_1 + \beta(\mathbf{x}), \qquad -1 \leqslant \phi_1 \leqslant 1$$

with $\gamma(\mathbf{x}) = \alpha \times \beta \neq 0$. Then, if

$$\gamma \cdot \operatorname{curl} \gamma = 0$$

a known result of classical analysis tells us that $\mathscr{E}$ is covered by a family of surfaces such that everywhere the vectors α and β lie in the tangent plane. Then $\mathbf{x}$ must always remain in the same one of these surfaces from which it started. We can use this surface for $\mathscr{E}$.

We will say a vectogram is *convex* if, whenever $v_1, \ldots, v_k$ belong to it, so does $\Sigma_{i=1}^k C_i v_i$, where $C_i \geqslant 0$, $\Sigma_{i=1}^k C_i = 1$.

Less trivial than the foregoing is the

CONVEXITY ASSUMPTION. All ϕ- and ψ-vectograms are convex.

Should this assumption be violated, there may be no solution. We do not reject the game but replace it by another in which the ϕ- and ψ-vectograms are the convex hulls of those of the old; that is, the new vectograms

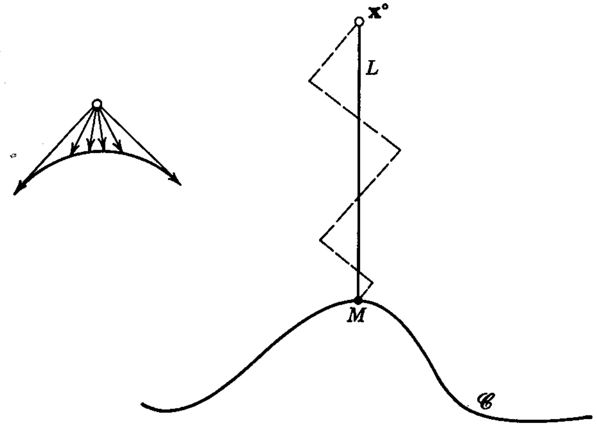


Figure 2.7.1

are the smallest convex ones which contain the old. If the new game can be solved, its solution will supply the essential information about the old.

An example will clarify matters. The reader will see how to apply its idea generally.

Example 2.7.2. In this game ψ does not appear, so that it really is a minimizing problem rather than a game. Here $\mathscr{E}$ is the part of plane above the curve $\mathscr{C}$ of Figure 2.7.1. The vectograms are the same for all $\mathbf{x}$; one is sketched. Let M be a high point of $\mathscr{C}$; P is to start from $\mathbf{x}^0$, directly above M, and reach $\mathscr{C}$ in the least time. Clearly a solution will entail a zigzag path arising from an alternate use of P's two extreme velocities. There will be many solutions.

Example 2.7.3. Let L be a vertical line through M. We alter matters by letting the vectograms preserve their form but letting them decrease in magnitude with the distance from L. Then clearly P does the better with finer zigzags which stay closer to L. There is now no solution.[15]

Perform the alteration described above, replacing the vectograms by their convex hulls. The game now has a solution: P traverses L to M. We see in what sense this solution is approximated by those of the unaltered game. Thus by replacing vectograms, when necessary, by the convex hulls we obviate troubles of the above type. It is easy then to interpret the original game.

If ϕ corresponds to a member of a ϕ-vectogram, then $c\phi$, where $0 < c < 1$, is a convex combination asserted to be also a member by the convexity

[15] The Value however exists.

assumption. But it turns out that such fractional multiples are seldom of importance in practical problems. If the motion of a craft is involved, decisions as to the best direction are the real problem; optimal strategies almost always dictate the maximal allowable speed. Therefore, in many of the problems to come, the vectograms shall not be convex in this fractional multiple sense (the reader can always make them so by introducing a new control variable like the c above, but he will find its optimal value to be 1), but only in the sense of closure with respect to vectors of distinct directions.

Somewhat similarly we have the

CLOSURE ASSUMPTION. All ϕ- and ψ-vectograms are closed.

The grounds are similar to those of the last assumption. If there were a convergent sequence of members of a vectogram that incurred increasingly favorable payoffs, technically the game might lack a solution. Therefore it is wiser to include the limiting velocity, which we presume to be optimal, in the vectogram. Thus we circumvent difficulties of this kind.

As before, when faced with a game with improper vectograms, here meaning they are not closed, we simply adjoin their closures and proceed with the analysis.

We may consequently assume that the lengths of the vectors in any vectogram are bounded, for, if not, the closure assumption would imply the existence of infinitely long vectors. If these were actually used in an optimal strategy, we could soundly judge the game to be pathological or trivial. If they were not, we could excise them and some neighbors from the vectogram without effecting anything essential.

The latter two statements together imply

We may assume all ϕ- and ψ-vectograms are closed and bounded and therefore compact.

We come now to the useful result:

The constraints on the control variables may be taken as constant.

That is, for example,

$$a_i \leqslant \phi_i \leqslant b_i \tag{2.7.1}$$

with a_i, b_i independent of $\mathbf{x}$. For we know that we may assume each vectogram to be a compact, connected (because of the convexity) set, and, from our general hypothesis as to the f_i, these sets vary smoothly with $\mathbf{x}$. We can then find a smooth mapping of these sets of vectors $\{\phi_1, \ldots, \phi_\lambda\}$ into, say the unit cube of Euclidian λ-space, which is also a smooth function of $\mathbf{x}$. Using the coordinates of the cube as new ϕ_i we will have

(2.7.1) (with $a_i = 0, b_i = 1$, but the values do not matter). The ψ-vectograms can be treated similarly.

In a game with terminal payoff, as only the location of $\mathbf{x}$ at termination matters, a change of time scale makes no effective difference. We can even make this scale change locally, varying it from point to point in $\mathscr{E}$. Formally this is accomplished by multiplying the right sides of the kinematic equations all by the same positive function of $\mathbf{x}$, that is, the typical KE is replaced by

$$\dot{x}_i = u(\mathbf{x}) f_i(\mathbf{x}, \phi, \psi).$$

It is clear that as long as $u(\mathbf{x}) > 0$, the curves of the paths are unchanged and hence the same strategies yield the same payoffs as before.

As we may assume the vectograms bounded, we can take $u(\mathbf{x})$ such that at each $\mathbf{x}$ its product with the longest member of the full vectogram is bounded throughout $\mathscr{E}$. Thus we may assert

In a game with terminal payoff we can arrange, without effectively changing the game, that the vectograms are uniformly bounded throughout $\mathscr{E}$.

2.8. A LEMMA ON CIRCULAR VECTOGRAMS

The following simple result will have utility in many of our later problems.

LEMMA 2.8.1. Let u, v be any two numbers such that

$$\rho = \sqrt{u^2 + v^2} > 0.$$

Then

$$\max_{\phi} \left[\min_{\phi}\right] (u \cos \phi + v \sin \phi)$$

is furnished by $\bar{\phi}$, where

$$\cos \bar{\phi} = +[-]\frac{u}{\rho}, \qquad \sin \bar{\phi} = +[-]\frac{v}{\rho}$$

and the max [min] itself is

$$+[-]\rho.$$

Proof. The quantity in round parentheses is the inner (scalar, dot) product of the vectors (u, v) and $(\cos \phi, \sin \phi)$ and therefore is the projection of (u, v) on the ray through the origin having inclination ϕ with the u-axis. The maximum then occurs when the ray lies along (u, v) and the minimum when its direction is precisely the opposite. Such correspond to the $\bar{\phi}$ asserted. The maximum [minimum] is the length [— the length] of the vector (u, v), which is $\rho[-\rho]$.

CHAPTER 3

Discrete Differential Games

3.1. INTRODUCTION

Like many problems of mathematical analysis, differential games can be quantized into discrete models. The smooth, continuous processes can be replaced by sequences of individual steps or moves.

One objective is the application of numerical methods by successive calculations. With an automatic computer at hand, this may be a tempting road to the solution, especially when the analysis is difficult. But generality is lost; we cannot see how the answer depends on the starting conditions and on the various descriptive parameters without calculating a vast profusion of cases. Moreover, as some of the following examples will show, many mathematical figures, such as singular surfaces or even the uniqueness of the optimal strategies, may be obscured or lost.

We do not stress this aspect here; in particular, we do not touch the question of convergence: proofs, that with a finer mesh, a discrete solution approaches the continuous one.

Our aim is rather intellectual adumbration. At this point discrete games can motivate and clarify many of the ideas to come. In the next section we show that even the general two-player zero-sum game with full information bears parallels to our theory.

We then go on to examples which are more literal quantifications of differential games. In Section 3.3 we present a battle game where each player strives to annihilate his opponent's forces. It might be interpreted militarily or, say, as two commercial firms each striving to force his competitor out of business. We use it both to illustrate a game of degree,

where the payoff is the force surviving on the victorious side or a game of kind, where the objective is simply extermination. Continuous versions of such games will be easy for the reader who has mastered the ideas in subsequent chapters.

Two pursuit games follow. The first, that of the hamstrung squad car, actually is better adapted to the discrete rather than the continuous format. The next is the ubiquitous homicidal chauffeur game; certain of its fine points come to light even here. The failure of others to do so is typical of the limitations of discrete methods. The final section sketches a step-by-step technique which requires only a partial quantization.

The veteran mathematician may skip these examples without loss of literal instruction. They contain no material specially required for later use; their purpose is a preview, possible because the difficulties of analysis are sidestepped. On the other hand, the less mathematically sophisticated reader will find in this chapter the spirit of our context, through analogues of ideas and parallels of nomenclature, even if he reads no further. However, the general discussion of the solutions in Section 3.2 is basic to the philosophy of game theory.

Finally, let us note that it is sometimes advisable to reverse the procedures of this chapter, that is, when confronted with a game with discrete moves yet with a certain logical coherence,[1] we might gain by replacing it by a continuous model. Such is tacitly done, for instance, in the later Examples 5.4 and 11.9.

3.2. THE GENERAL DISCRETE GAME

Our study in this book is of two-player zero-sum games with complete information. Any such discrete game can be diagrammed—the so-called extensive form—as the sample shown in Figure 3.2.1. Rumination will lead to some instructive parallels with differential games.

Each position is represented by a circle or rectangle and from each it is possible to go to certain others along the connecting lines in a *downward* direction. Small circles indicate that it is P's move, that is, with the minimizing player rests the decision as to which subsequent position. The squares pertain similarly to E. Thus the topmost circle is the starting position; it is P's move and he has three choices, all of which give the next option to E.

We can think of a counter being placed at the starting position and the partie consisting of its successive moves. The occupied circle or square is the discrete counterpart of the state $\mathbf{x}$ in a differential game. Ultimately the counter or $\mathbf{x}$ reaches one of the flat rectangles which represent terminal

[1] The exact type is difficult to specify. We discuss this point later in the chapter.

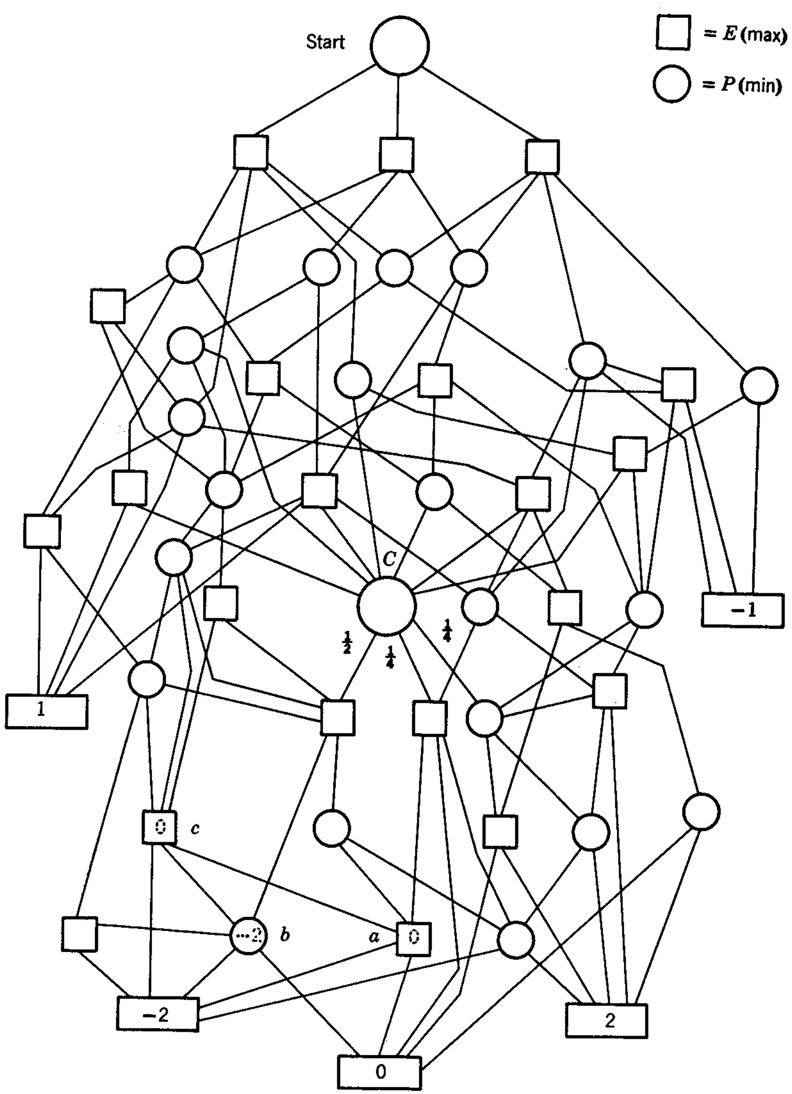


Figure 3.2.1

positions; play is over and the number in the rectangle is the payoff. These final positions correspond to the surface $\mathscr{C}$; it is clear that we have here a "terminal payoff." We learned in the last chapter that any differential game could be so formulated; we will see soon that discrete games can have other representations, and the present one is the particular type that exhibits terminal payoff.

Chance moves can appear; then the decision as to the next move is made by a random device with certain given probabilities rather than by P or E. We have included one such position; it is the large circle labeled C, and the probabilities of the three possible ensuing positions are marked on lines to them. This random element requires us to define the Value as the minimax of the *expected* payoff.

Let us now solve the game. From the position labeled a, E has two choices. Both lead to termination and the payoffs will be -2 and 0. His goal being the maximal payoff, he will clearly prefer the latter; let us label this square with the number 0. From the position b, P has the same two choices, but he will select -2 and this is marked in the circle. From c, E has three choices; the payoffs being the -2, -2, 0 as marked; the last is max and so square c is marked 0. In this way we work upward from the terminal states until finally the starting position is labeled; this label will be the Value of the game. The optimal strategies are represented by the lines that lead to positions marked with the Value.

The chance state C is, of course, an exception. When its three ensuing positions have been numbered, we write in C their expectation, here a linear combination of these numbers with coefficients, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$.

Exercise 3.2.1. Complete the solution and show that the Value of this game is $-3/2$.

Exercise 3.2.2. Start to construct a diagram in the manner of Figure 3.2.1 for the game of tic-tac-toe (tit-tat-toe, oughts and crosses). We say "start," for the full task will be tedious and long, but we wish to emphasize exactly this point.

We will now glean some insights.

1. The numbers that were marked in each position constitute the analogue of the basic function in differential games which bears the term $V(\mathbf{x})$.

Observe that each position in the diagram can be regarded as the starting point of a subgame. Retaining only positions which can be subsequent to the given one by means of some series of moves starting from there, and erasing the rest, we have just the figure relevant to this subgame. In other words, the subgame is what follows if $\mathbf{x}$ were placed on the arbitrary given position and the players began playing from there.

Similarly, in differential games, $V(\mathbf{x})$ is the Value of the game that results if a partie commences from the point $\mathbf{x}$.

2. The optimal strategies are intimately and simply related to $V(\mathbf{x})$. When $E[P]$ makes a decision as to his move from a position, clearly his best choice is that which maximizes [minimizes] $V(\mathbf{x}^i)$ where $\mathbf{x}^i$ ranges over the positions immediately ensuant to $\mathbf{x}$.

A continuous version of this idea dominates what we will soon call the main equation in differential games. It is a first-order partial differential equation in $V(\mathbf{x})$ and is based on the limit of the transition between neighboring $\mathbf{x}$ which renders minimax V on the later one.

The route of $\mathbf{x}$ in the discrete game, when play is optimal, corresponds to the optimal paths in a differential game. The latter satisfy a system of ordinary differential equations and are characteristics of the main equation.

3. Our procedure in solving the discrete game was to start at the terminal position and work backward, obtaining successive evaluations of $V(\mathbf{x})$. It is hard to conceive of any alternate principle, for at the outset V is known only at the terminal states.

There is a counterpart here too in differential games. We will use the terminal surface $\mathscr{C}$ as a seat of initial conditions to integrate the aforementioned differential equations for the optimal paths in $\mathscr{E}$. Such will motivate our reversal of time and the retrogressive form in which we will later obtain our integrals.

4. Our last point is much more general and more difficult to express precisely. An analysis of a game such as is portrayed in Figure 3.2.1 can hardly be systematic; there appears no other recourse than proceeding one position at a time to exhaustion. The importance of this lack of pattern becomes manifest if we imagine the figure vastly enlarged.

If the reader has attempted Exercise 3.2.2 he will see that, even for a game of childish simplicity, the diagram, even though manageable with sufficient diligence, is already formidable. What would it be for a game such as chess? Each position essentially corresponds to a permissible arrangement of the chess pieces on the board[2] and the connecting lines to all possible legitimate moves of the appropriate player. The magnitude of the diagram would be astronomical, that of the solution easily beyond the scope of the mightiest of present-day computers.

On the other hand, consider the game diagrammed in Figure 3.2.2. The regularity is evident. A pattern of answers emerges concurrently with the act of solution. In fact, we can easily prove by induction on n that the

[2] Certain auxiliary information is requisite too. For example, two states of a game with identical piece arrangement, but in only one a forfeiture of the castling privilege through prior moves, would have to count as two distinct positions.

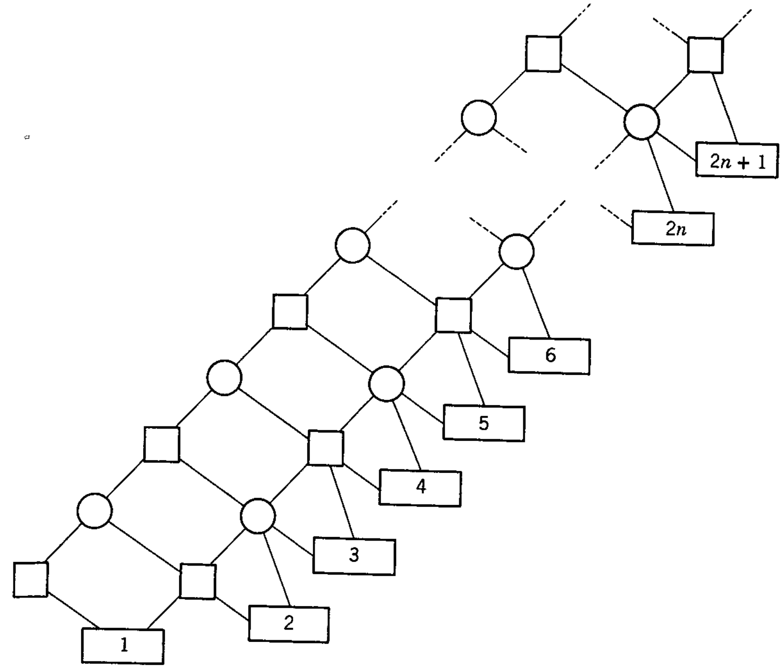


Figure 3.2.2

values of V in the uppermost four boxes are

$$\begin{matrix} & 2n & & 2n+1 \\ 2n-1 & & 2n. & \end{matrix}$$

Yet it is obvious that a segment of this diagram, capped at the top with some simple configuration that supplies the requisite opening move, would, if n were sufficiently large, be of greater magnitude than that of chess.[3]

The distinction is one of pattern—of inherent logical structure. The latter game yields to the techniques of difference equations or recurrence relations because each position is related to its neighbors with a certain uniformity which is lacking in games of the chess type.

But such uniformity is the very stuff of mathematics. A planetary orbit is calculable in principle just because at every instant the same law of Newton holds. Stock market prediction is not because we cannot find a like logical system governing it.

[3] The reader should not be misled by the long, narrow configuration of Figure 3.2.2 which makes it appear "one dimensional," whereas 3.2.1 seems more "two dimensional." This is beside the point; it is possible to construct "multidimensional" versions of the former figure relating to games similarly solvable by systematic recurrence.

We hope the spirit of these ideas is clear; a sharp delineation is difficult. Our statement that chess does not possess the type of logical structure rendering it amenable to mathematical analysis does not imply that chess is an erratic, illogical game.[4] Rather it means that analysis apparently can but construct piecemeal chains of cause and effect; the mathematician appears to be able to do no more than emulate the deductions of a competent player.

The quintessence of differential games is not its use of the classic tools of analyses, as the term "differential" might appear to imply, but its real concern is with games with an inner logical structure. The analytic techniques, such as differential equations, might be replaced by discrete but systematic methods once the continuous game is replaced by an appropriately quantitized model.[5] In either case, there is a connection between contiguous states that maintains a uniformity of relationships throughout a full phase, if not all of, a partie. Such is what makes the theory possible.

3.3. BATTLES OF EXTINCTION

In the simplest form of these games there are but two state variables, x and y, which are the forces of the two antagonists. The game is over when either is reduced to zero, the payoff to the survivor being the amount of his own force remaining.

For $\mathscr{E}$ we take the first quadrant of the x, y-plane; $\mathscr{C}$ will be the union of the positive x- and y-axes. If y pertains to the maximizing player E, then the H will be $-x$ on the y-half axis and y on the x-half axis.

The options of the players, as expressed by the "kinematic equations," should be, say, between moves that bring the partie to termination more rapidly or deplete the enemy faster. The effects should be greater with his own greater or the enemy's lesser strength.

A good simulation of any aspect of reality is hard to achieve with only two state variables and the above kind of moves. We do not claim any realistic prototype for the example below. But for a simple illustration of the discrete method, we must keep the dimension of $\mathscr{E}$ low to avoid a superabundance of positions. Such would be unnecessary, of course, if the model were solved by an automatic computer.

In the following example we have what might well be a quantized version of a differential game as just described. It is not diagramed by

[4] Or that it will not yield to techniques other than those of traditional mathematics such as self-learning computers.

[5] Such models are sometimes closer to reality, as the aforementioned Example 5.4, War of Attribution and Attack. In such cases, the advantage of the "differential" approach may be rightly viewed as one of efficacy rather than versimilitude.

the scheme used in the last section, but its format is closer to our standard one of a differential game with terminal payoff.

Example 3.3.1. A simple battle of extermination. The playing space $\mathscr{E}$ is the set of vertices of the reticulated first quadrant of Figure 3.3.1 with the standard x, y coordinates. Its left and lower boundary points form $\mathscr{C}$ (where one or the other forces is zero as discussed above); the values of H are as marked. The vectograms are as sketched around the border. For example, if $y = 0, 1, 2,$ or 3, E may make either of the two "knight's moves" as shown in the lower left vectogram; for $y = 4, 5, 6,$ or 7, he has the choice of the three as depicted immediately above; for $y \geqslant 8$, he employs the uppermost vectogram.

Moves are made alternately. We think of a counter **x** placed on any lattice point (for as in the continuous cases, each point of $\mathscr{E}$ may serve as a starting position). The players move alternately by selecting one of the displacements of **x** allowed by the appropriate local vectogram. The partie ends when **x** reaches or crosses $\mathscr{C}$. The payoff is the value of H at the point nearest to the point where the move, as represented by a straight arrow, crosses $\mathscr{C}$. A midway crossing scores the higher in absolute value of the two adjacent points.

We solve this game, as before, by finding $V(x, y)$, the Value, if x, y is the starting point. But a minor difference from the general case intervenes:

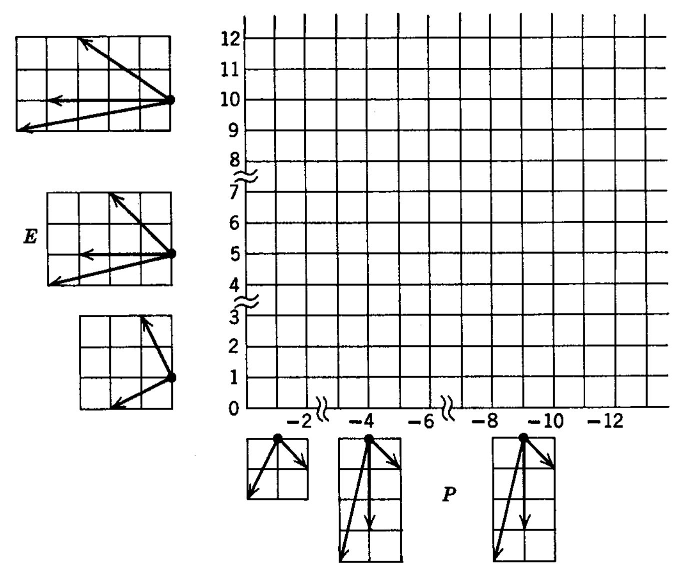


Figure 3.3.1

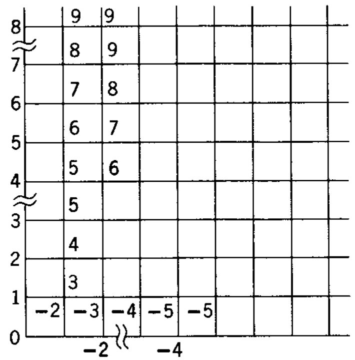


Figure 3.3.2

either player may have first move. Thus two distinct V are required. In our subsequent diagrams we shall write the V, when E has the move, above (and to the right) of a vertex and for P, below.

We begin by locating those points from which termination must occur after one move, regardless of the player's choice. We can write in the V for the moving player, as it is his best possible outcome. The result is as in Figure 3.3.2. For example, at the point $(1, -1)$,[6] the two moves available to E lead to meeting $\mathscr{C}$ at $(3, 0)$ and crossing it at $(\frac{1}{2}, 0)$; the two payoffs are 3 and 1; E picks the max $= 3$ and we write it over the vertex. On the other hand, P's two choices lead to payoffs of -1 and -2; we write the latter, the min, under the vertex. We can fill in nothing yet on a vertex such as $(3, -2)$, for both of E's moves do not reach $\mathscr{C}$, but from $(4, -2)$ all three possibilities do and the max is 6.

This step is an instance of the general procedure, which is governed by the rule:

When from a point $\mathbf{x}$, *the* V *relative to his opponent is known at all points to which a player may move, his* V *at* $\mathbf{x}$ *is the max (if he is E) or min of these* V.[7]

Successive application of this rule determines fully the two V. For example, if P is to move from $(5, -1)$, his move carries him to $(3, 0)$, where $V = H = 3$, or to $(4, -2)$, where V (for E) $= 6$; the min 3 is thus his V. After $(5, -1)$ is so labeled, both outcomes of a move for E from $(3, -2)$ are known and it may be treated similarly. Proceeding thus leads to the results in Figure 3.3.3.

[6] We will use the values of H as coordinates.

[7] Of course, on $\mathscr{C}$, H counts as V for both players.

The optimal strategies are known at once from the two V; a player makes the decision implied by our rule. Figure 3.3.4 depicts two instances of optimal play with the adjacent starting points $(11, -14)$ and $(10, -14)$ at which V (for E) $= 4$ and -3. The solid arrows denote the moves of E; the broken ones, those of P. If at a point a player has more than one optimal move, all are shown; thus the figures include all possible optimal strategies from the selected initial points.

There is one difficulty, minor in this case, but perhaps more serious in possible variants, which we leave to the reader as

Problem 3.3.1. Observe that in E's lowermost (Figure 3.3.1) vectogram and P's rightmost one, there is a pair of translations equal and opposite, namely, one space horizontally and two vertically. For some parts of $\mathscr{E}$, where both these choices are admissible, a possible outcome is that each player perseveres in these choices so that $\mathbf{x}$ oscillates between two points and the partie never terminates.

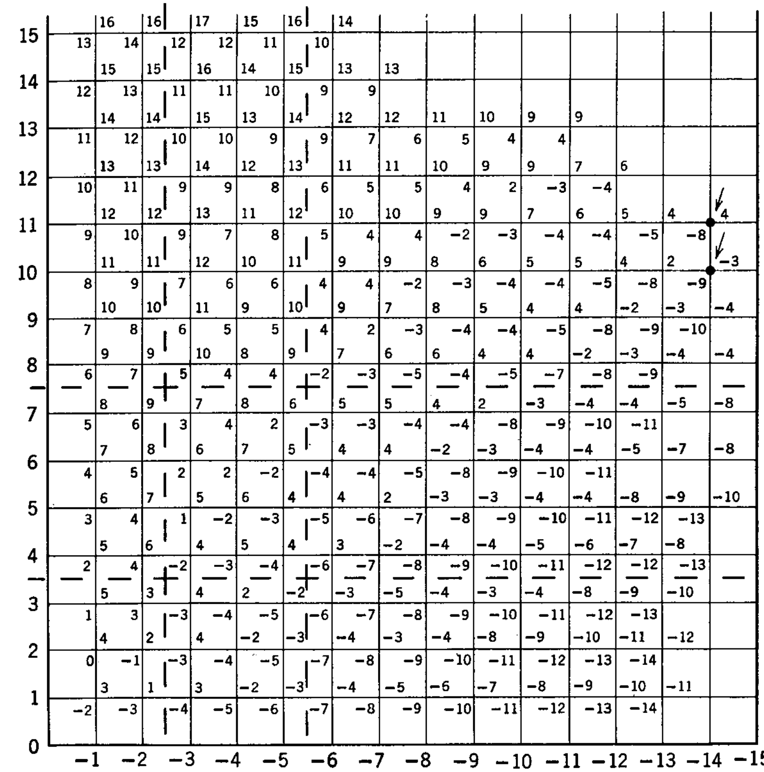


Figure 3.3.3

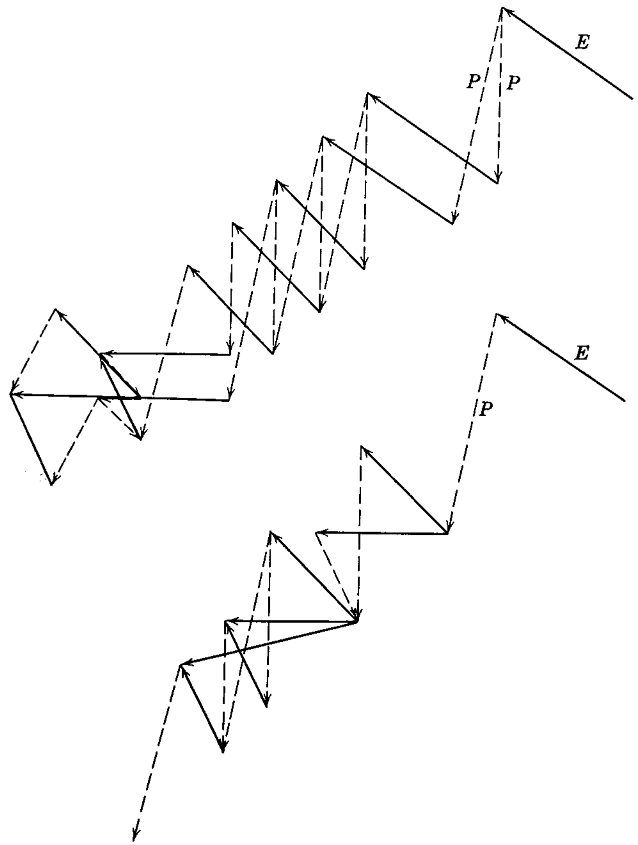


Figure 3.3.4

The rules must be amended. Let us do so simply by assigning 0 as the payoff of never ending play.

Show, then, that optimal play will always end and show how to find the V at the critical pairs of points.

Investigate the matter generally: find criteria for such nonterminating oscillations to be optimal.

Problem 3.3.2. Note that in the upper chart of Figure 3.3.4, P often has choices of optimal moves, while E's is unique. In the lower chart, on the other hand, E appears to have the greater freedom.

Discuss, in a rough way, what factors govern the amount of choice allowed a player under an optimal strategy.

Suppose we were not interested in the number of survivors but simply wished to know which player—when striving for the contrary end, of course—becomes extinct first. That is, we have a *game of kind* in which the player who exterminates his opponent is winner.[8]

Of course, the solution just obtained includes that of this new game; we have but to note the sign of V and otherwise disregard its value. But is there not a more direct method which does not require coverage of all of $\mathscr{E}$? There is, and as it adumbrates the ideas of Chapters 8 and 9 on games of kind, we shall look into it.

For definiteness we shall suppose E to move first. Then there is a set of starting points from which he can win—force $\mathbf{x}$ to the positive y-axis—and another from which he can be beaten. We should expect these two sets to be separated by a third set for which the outcome is a draw—$\mathbf{x}$ reaches the origin under optimal play. If we knew this third set, then, the problem would be solved. We can expect it to be sparser than the other two; in fact, it ought to be a slender array whose configuration resembles a curve through the origin. It is the counterpart of what later, in the continuous case, will be called the *barrier*.

Example 3.3.2. The battle of extinction: game of kind. This game, just described, has the solution shown in Figure 3.3.5, where the indicated vertices constitute the barrier.

To each encircled point (other than the origin) of the barrier there is attached one or more arrows indicating E's optimal strategy.[9] The reader can verify that, if E makes any such move, then, whatever move P may make in response, $\mathbf{x}$ will not be brought below the barrier, nor can it reach the x-axis, save possibly at the origin. Suppose P brings $\mathbf{x}$ back to the barrier; E replies with another indicated move. As long as this alternation persists, as the reader will easily see, $\mathbf{x}$ will be forced leftward. Should a move of P deliver $\mathbf{x}$ to a point above the barrier, then E can keep $\mathbf{x}$ from going below and compel it leftward, if anything, even the more strongly. Thus $\mathbf{x}$ must reach the y-axis; the outcome will be either a win for E or a draw.

Further, the barrier is the lowermost "curve" with the above property. This remark illumines the construction: suppose barrier points have been constructed on the lines where $x = 0, 1, \ldots, n$. To construct the next, with $x = n + 1$, we test the points on this line for the requisite property starting at the bottom and working upward, trying each eligible move of E

[8] Checkers is essentially an example of such a recreational game.

[9] In Chapter 8, we shall learn that in games of kind optimal strategies are defined only at points of the barrier.

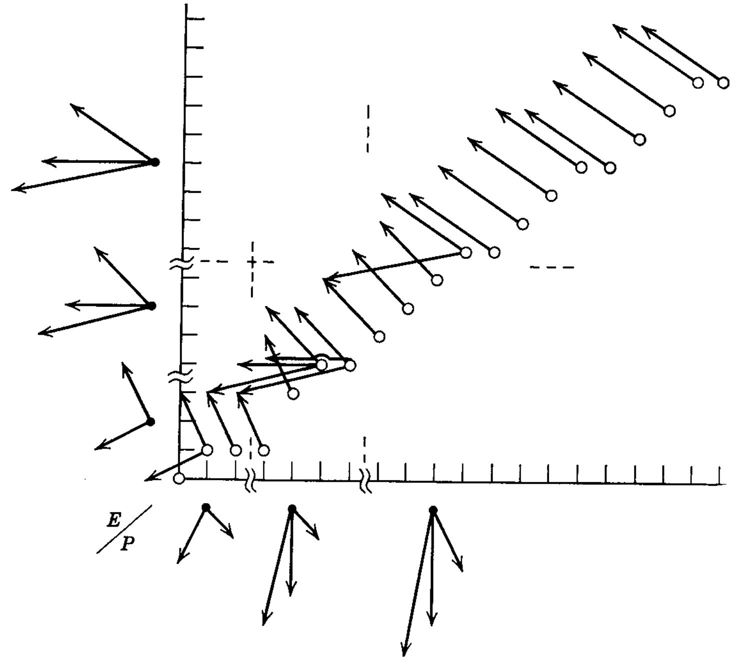


Figure 3.3.5

in each case.[10] The first that passes the test is our new barrier point. The procedure begins, of course, with the origin.

This construction makes clear the asserted lowermost property. In turn, this implies that if the starting point is below the barrier, P can force a win.[11]

Exercise 3.3.1. Construct the barrier if P has the initial move.

3.4. TWO DISCRETE PURSUIT GAMES

As the last example, these two will mimic the format of continuous differential games. They will also illustrate the compactness gained by using a reduced space.

The first game, even when removed from context, appears as a neat and charming bagatelle.

[10] In practice, we need not be this pedantically thorough; it is clear that no barrier point can be lower than its left neighbor.

[11] Note that the barrier lacks the purity promised in the preliminary discussion. It is not comprised entirely of points for which the outcome is a draw but contains many which win for E. Such does not happen in the continuous case.

Example 3.4.1. The game of the hamstrung squad car. The squad car P is chasing a car E of criminals through a city whose streets form a perfect, unbounded square lattice. Even though P has twice the speed of his quarry he must obey the municipal traffic rules, which prohibit left and U-turns, statutes which E disregards.

In our quantized version the players alternately perform discrete moves. If E is at the vertex E of Figure 3.4.1, on his turn he may move to any of the four adjacent vertices of the lattice as shown. Suppose P is at the point O and has come from C on his previous move. On his turn he may either move two spaces forward to A or execute a right turn to B. Capture occurs when P and E coincide or are adjacent, that is, if P is at O, E is captured if he is any of the nine points labeled ×. Finally P has the first move, and the payoff is the number of moves of P until capture.

Let us adopt a reduced space. We take O, P's position, as origin of discrete rectangular coordinates with the vector CO of his previous move pointing along the y-axis. A position is then described by a point $\mathbf{x} = (x, y)$, the latter being E's coordinates relative to P. We may think of a full-scale map of the city attached to the roof of the squad car, with $\mathbf{x}$ as the point of this map immediately over E.

The disadvantage of a reduced space is that moves generally become more complicated.[12] Suppose that P moves two spaces forward to A; such is equivalent to moving $\mathbf{x}$ (as on the figure) back two spaces to A'. Now let P turn right to B. Orienting ourselves along the vector OB, we

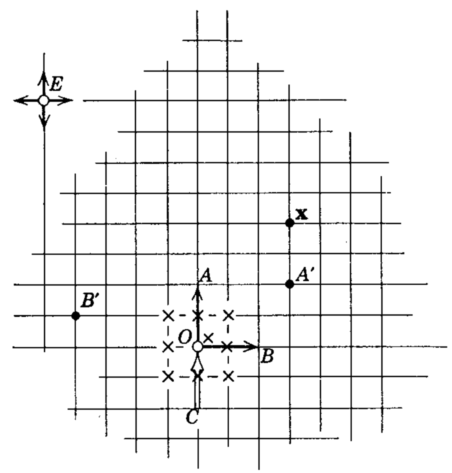


Figure 3.4.1

[12] In the continuous case, it is the KE which suffer similarly.

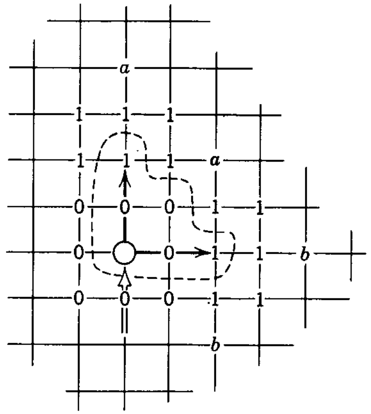


Figure 3.4.2

see that the **x** illustrated is four spaces to the left and one forward of B. Thus the effect of the move is to transfer **x** to B', where these coordinates hold in the original scheme.

Now for the solution! Again we achieve it by successively calculating $V(\mathbf{x})$ from the capture region outward. We will label the vertices with their values of V as we go.

We begin by marking the origin and its eight neighbors 0, for at these points P captures with 0 moves. Inasmuch as he moves first, we can immediately spot the points where $V = 1$; they are marked on Figure 3.4.2.

From here on, the process assumes generality. The next step is to delineate the points hemmed in, from E's viewpoint, by 1's or 0's, that is, the points **x** such that if E were at **x** all four of the points to which he could move have already been marked, at least one of them with a 1. These points are enclosed by the broken line in the figure. Next we spot the points not previously marked such that a move by P will bring them to one of the enclosed positions. There are two such points (a) if P moves forward and two (b) if P turns right. The four points, a and b, are labeled 2.

Let us verify the last step. If **x** is at one of the points $a[b]$, P can bring **x** to the enclosed set by moving straight [turning]. (Such moves are part of his optimal strategy.) It is now E's turn; his optimal strategy demands that he shift **x** to a point with the highest possible label, which here is 1.[13] It is now P's move, and we already know (note the induction!) that he can capture **x** on such a position in one move, making a total of two.

[13] It is even better for E to move to an unlabeled point if he can. Our construction makes this impossible here.

The ensuing steps are similar. Suppose the points with $V = 0, 1, \ldots, n$ have all been found and marked. Let S consist of the points **x** such that all four neighbors of **x** are marked, with at least one mark an n. The points, if any, not already marked and such that one move of P brings them to S, are now marked $n + 1$.

The reader who executes this construction for himself will enjoy a mild and diverting task. When $V = 11$, he will have constructed Figure 3.4.3. He will find that no further steps are possible; the configuration is complete.

If the game is started from an unmarked point, E can permanently evade capture. (What is his strategy?)

Research Problem 3.4.1. If we increase P's "speed" by letting him move 3 or more spaces on a move instead of 2, is it still true that there are starting points of permanent escape for E? The capture region should similarly be enlarged to eschew the possibility that P pass over E without capturing him.

Exercise 3.4.1. Chart some trajectories of optimal play in the realistic space.

Example 3.4.2. The homicidal chauffeur game: a discrete version. Our discrete version of this game will be played on a triangular lattice, but in most other respects the ideas of the last example persist. Again moves are made alternately, with P going first; he moves two spaces at a turn, while E moves one.

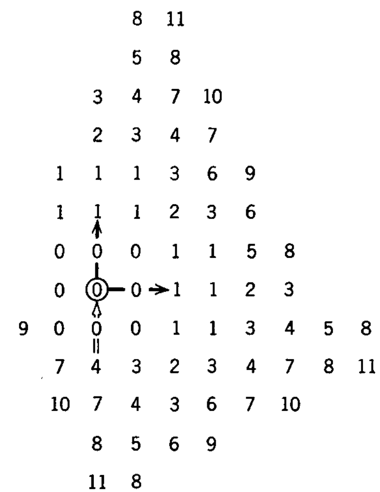


Figure 3.4.3

If P is at O in Figure 3.4.4 and has come from C on his previous move, then he must choose between the three moves which take him to A, B, or D. In this way we have a simulation of a fast pursuer subject to a limited curvature of path. On the other hand, E may move one space to any of the six adjacent points as shown.

Capture occurs, if P is at O, when E occupies O or any of the six contiguous points, all labeled $\times$ in the figure. Again the payoff is the number of moves of P until capture.

Once more we employ a reduced space in which P is at the origin O, his prior vector CO being on the vertical axis, and $\mathbf{x}$ is E's affix in this coordinate system. We leave the reader to apprehend the displacements of $\mathbf{x}$ in response to the three possible moves of P.

The technique of solution is the same as that of the previous example: we determine V starting from the capture region (the $\times$) and working outward. The points where $V = 0$ or 1 can be located immediately. As before, when we know the points where $V = 0, \ldots, n$, we first locate the set S of $\mathbf{x}$ such that if V is at $\mathbf{x}$, for all six contiguous points $V \leqslant n$ and max $V = n$. Then the set where $V = n + 1$ are those points to which no V has been previously assigned and, for some one move of P, can be brought to S.

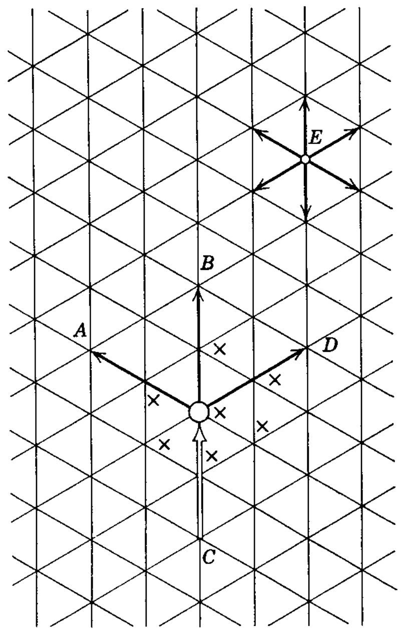


Figure 3.4.4

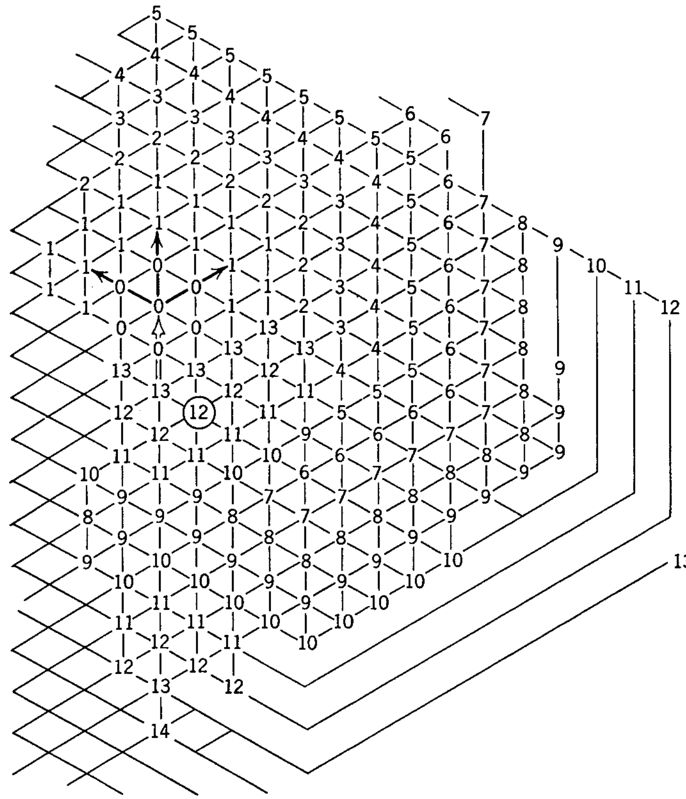


Figure 3.4.5

What results is shown in Figure 3.4.5. This time all lattice points are marked; P can always capture. Because of symmetry, one side is omitted. Observe the successive values of V. Near the top of the chart they occur consecutively in orderly concentric rows. The ends of each row curl around that of its predecessor, and this happens more and more until $V = 9$. The row of 9's extends right across the bottom of the chart leaving a cavity near the tail of the double arrow, which is filled by V's ranging from 10 to 13. These entries consist of the starting points that lead to optimal play with a swerve (Section 1.5).

An instance in the realistic space, starting from the encircled point where $V = 12$, is plotted in Figure 3.4.6. Note how P must go twice leftward before veering back to get at E; note how E at first follows in an effort to intensify P's need for this maneuver.

This is not the only outcome, for in this game the optimal strategies are often not unique. The cause is merely the quantization and is easy to apprehend. Suppose, say, E wished to travel a long stretch horizontally (in terms of the orientation of the figures) as rapidly as possible. He would have to zig-zag and at alternate points would have two equally good moves (he could go slantwise up, then down, or down-up).

For reasons such as the above many of the niceties of the homicidal chauffeur game become blurred in its discrete version. In later chapters we shall study a curve called the *barrier* which delineates the starting points leading to a swerve. On the barrier V is discontinuous. The counterpart can be crudely seen in Figure 3.4.5 by looking for pairs of adjacent points at which V differs by more than 1.

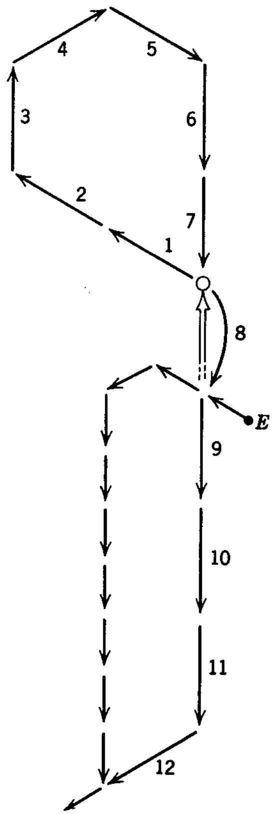


Figure 3.4.6

3.5. QUASI-DISCRETE GAMES

An occasional use of discrete models is that sometimes, when we are perplexed about a differential game or some phenomenon therein, we can gain a foretaste of the truth by a step-by-step solution of a discrete analogue. Sometimes better for this purpose are models that are partially discrete, partially continuous. We will sketch lightly one such possibility.

Let us retain the continuous vectograms of a differential game but decompose time into a series of fixed (and usually equal) intervals. The players move alternately. A move lasts an interval and during it the control variables are kept fixed.

We will again use the homicidal chauffeur game. Following the pattern of the last example, we take vectograms[14] as in (a) of Figure 3.5.1. On a move a player makes a displacement of a prescribed magnitude, but he has a choice over a continuum of directions.

The stepwise construction of V begins as shown at (b) of the figure.

[14] We are not following exactly the dictates of the previous paragraph in that we are not basing our work on the ϕ-vectograms of the continuous version in the reduced space. Such vectograms will be described in Section 10.2.

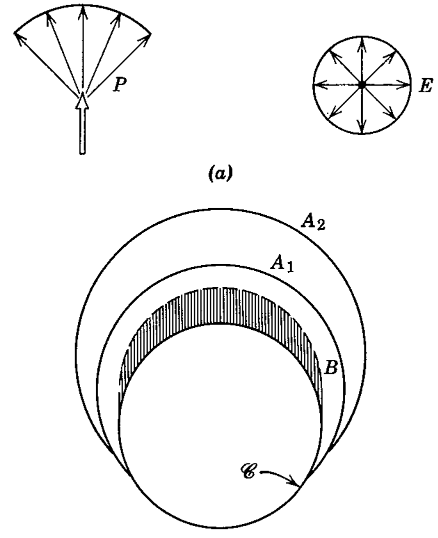


Figure 3.5.1

Using the same reduced space as before, the capture region will be, say, the circle $\mathscr{C}$. The first step to find the subset S_1 of $\mathscr{E}$ of points $\mathbf{x}$ such that one move by P will bring $\mathbf{x}$ to or within $\mathscr{C}$. The set S_1 will be bounded underneath by part of $\mathscr{C}$ and above by curve such as A_1 in the figure; it is just the set where $V = 1$. Next we find the subset of S_1 which inhibits escape by E, that is, the points $\mathbf{x}$ of S_1 such that no move by E can cause $\mathbf{x}$ to leave S_1. In the sketch it lies below the curve B. The next cycle starts by finding S_2, the set of points not in $\mathscr{C}$ or S_1 such that some move by P will bring them to the above set between B and $\mathscr{C}$. Then S_2 is the set where $V = 2$. We continue so.

An actual execution of this procedure would seem to be extremely unwieldy if n exceeded 2. For a planar case as described, a practical method is to use two sheets of tracing paper, one for the A curves, one for the B, and alternately trace from one to the other.

CHAPTER 4

The Basic Mathematics and the Solution Technique in the Small

The fundamental mathematical concepts that will permeate the rest of this book appear here.[1] The main equation, a first-order partial differential equation for the Value, $V(\mathbf{x})$, is derived by two methods. Its appropriate integrals can be shown actually to be the Value of the game in each particular case by what is termed the verification theorem. The theorem is proved and its use illustrated by a varied sequence of examples.

The final sections construct our standard method for obtaining solutions in the small, that is, in all cases where there is no interference from singular manifolds.

4.1. THE NATURE OF A SOLUTION

When a particular differential game has been solved, the results will generally embody such entities as

1. The Value: the function $V(\mathbf{x})$ defined over $\mathscr{E}$.

2. The optimal strategies: (vector) functions $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ defined over $\mathscr{E}$. They may or may not be unique. In the latter event we might be interested in obtaining the complete set of all optimal strategies or we might be satisfied with just one.

3. The optimal paths: when either the optimal strategies are unique or, for some reason, we have decided to devote our attention to a unique choice, there will be a set of optimal paths which are traversed by $\mathbf{x}$ when these strategies are played. These paths should fill $\mathscr{E}$ and each should terminate on $\mathscr{C}$.

[1] A re-edited version of RAND Report RM-1411 (21 December 1955).

There will be cases where these entities do not all exist, at least not in all of $\mathscr{E}$. The later Example 7.8.1 is an elementary instance. There are more subtle types; some counterexamples in works on the calculus of variations would be such instances of one-player games. In such cases we regard a solution as the supplying of as much enlightening information as the circumstances warrant.

We prefer not to hedge in the concept of a *solution* by too rigorous a definition. In the event of the nonexistence, in all or part of $\mathscr{E}$, of the entities 1, 2, 3 or some other pathological aspect, we will regard the game as solved when these phenomena are elucidated and understood. Nonexistence is not a calamity; usually there is a simple and enlightening explanation.

Even in completely nonpathological case, when 1, 2, 3 all exist without complications, it is not always necessary to specify explicitly all three. For example, if $V(\mathbf{x})$ is known, the optimal strategies, as we shall shortly see, can be computed as known functions of the state variables and the partial derivatives of V. If the optimal strategies are known, the optimal paths follow by integration of the kinematic equations; if V is integral, it becomes known through a functional elimination.

There will be cases where processes of the latter types are perfectly routine in principle but oppressively tedious to compute explicitly. Whether it is worth the trouble of doing so depends on what motivated the problem. If it was to illustrate ideas, as are many in this book, then often the fruits of the labor bring no further lucidity. Even with a practical problem it may be only certain aspects of the solution that will find application.

Thus, in the ensuing examples, in this regard there will be considerable variety, guided by judging the value in interest against the cost in labor, as to how detailed are the solutions presented.

To be purely logical, the solution should be expressed in terms of K-strategies. However, we will reserve this concept as an ancillary tool with which to give solutions a rigorous meaning when this is deemed necessary and to prove their validity. We shall demonstrate how such is done later in this chapter. In the future we will take the possibility for granted; we shall speak of strategies rather than tactics and generally think in terms of integrations of the kinematic equations.

We recall that the data of a game include a particular starting point in $\mathscr{E}$ and that we have used the term "game" rather freely for what should be a family of games. When we speak of the solution of a game for a certain subset $\mathscr{E}'$ of $\mathscr{E}$, we will refer to all games with starting points in $\mathscr{E}'$.

The process of solving a game splits into two phases. It generally turns out that the region $\mathscr{E}$ is to be divided into a number of parts separated by the surfaces which we shall later call *singular surfaces.*[2] In each part the solution will be smooth, that is, V will be of the class C_1. By this we mean that surfaces of constant V have continuously varying tangent planes; analytically, for all small vectors $\mathbf{u} = (u_1, \ldots, u_n)$

$$V(\mathbf{x} + \mathbf{u}) - V(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial V(\mathbf{x})}{\partial x_i} u_i + |\mathbf{u}|\, o(|\mathbf{u}|). \tag{4.1.1}$$

Similarly, except possibly on singular surfaces, the optimal strategies will be continuous functions denoted by $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ when they are unique; when they are not, we shall often assume such continuous functions can be selected.

On the singular surfaces a variety of special kinds of behavior can occur. This diversity has thus far defied a systematic theory which can be specialized to treat each type. A classification and notational scheme will be attempted in Section 6.1, but it does little more than catalogue the possibilities. Much of the space devoted to theory in this book concerns singular surfaces, and the theory of each type is distinct.

We shall at times use the term *in the small* to refer to the smooth parts of the solution found between the singular surfaces. The problem of identifying the singular surfaces and assemblying the smooth parts into the total solution will be described by the phrase *in the large.*

We shall see that the technique in the small is one of differential equations. This phase of the problem plays a part in full solution somewhat analogous to that played by the Euler equations in the calculus of variations. But we have been impelled to innovations of technique. Although our methods can handle the classical problems by viewing them as one-player games[3] (the second player is inactive in that his vectograms are null), we cannot do so through "extremals" but must distinguish minimum and maximum at the outset; how else could we handle games?

Not all types of singular surfaces are foreign to the classical calculus of variations.[4] But until we adopted the present game theoretical point of view, there was no compelling motive for identifying them, and their presence has been tacit. In our work, not only is the variety richer but the type of problem that game theory makes prominent emphasizes their importance.

It is hard to make a categorical statement as to the relative importance of the in the small or in the large phases. In some problems the integral solutions between the singular surfaces are simple, but the latter themselves are numerous, varied, and difficult. In others the integrals yield a rich family of paths which fill $\mathscr{E}$ with little or no singular behavior. The homicidal chauffeur game is an instance of the former; in the next chapter (Section 5.5) the game of the isotropic rocket, a variant of the same problem, falls in the latter category. The nearest we can come to a general criterion is to say that linear vectograms (the control variables appear lineally in the KE and G) imply many singular surfaces; indeed certain types can occur only in such linear cases.

[2] There may also be singular manifolds of dimension $< n - 1$; such of course cannot separate the components of $\mathscr{E}$. In the theory as thus far developed, they have been rather neglected. The term surface means an $(n - 1)$-manifold in n-space.

[3] See, for example, the dolichobrachistochrone problem (Section 5.2).

[4] See, for instance, Example 7.2.1.

4.2. THE MAIN EQUATION

We suppose that the Value of a differential game exists. It will depend on the starting point $\mathbf{x}$ and we denote it by $V(\mathbf{x})$. We shall show that $V(\mathbf{x})$ satisfies a first-order partial differential equation, to be called the *main equation* (abbreviated ME), whenever $V(\mathbf{x})$ is of class C_1. We shall write here and in the future

$$V_j \text{ for } \frac{\partial V}{\partial x_j}, \qquad j = 1, \ldots, n.$$

The ME is

$$\min_{\phi} \max_{\psi} \sum_{j} \left[V_j f_j(\mathbf{x}, \phi, \psi) + G(\mathbf{x}, \phi, \psi) \right] = 0. \tag{4.2.1}$$

Summations without limits, as written here, will be understood to run from 1 to n, where n is the dimension number of $\mathscr{E}$.

From the minimax assumption, it makes no difference if the min and max in (4.2.1) are reversed. It is understood that they range over all (vectorial) ϕ and ψ which satisfy the constraints.

We shall give two proofs. The first will follow immediately. Our treatment will be heuristic, inasmuch as we shall erect our serious foundations on the second proof, but the mathematician will easily see how to render it rigorous. The advantage of this first derivation is its straightforward, instructive character. It is the continuous counterpart of the process used for solving the discrete games of the preceding chapter.

We utilize what might be called the tenet of transition. The germ of the idea is that we are dealing with a family of games based on different starting points. Let us consider an interlude of time in midplay. At its commencement the path has reached some definite point of $\mathscr{E}$. We consider all possible $\mathbf{x}$ which may be reached at the end of the interlude for all possible choices of the control variables by both players. We suppose that, for each endpoint, the game beginning there has already been solved; in other words, V is known there. Then the payoff resulting from each choice, ϕ, ψ *during the interlude* will be known, and the control variables are to be so chosen as to render it minimax. When we let the duration of the interlude approach zero, the result yields a differential equation.

We put the above reasoning formally. Let V be known at $\mathbf{x}$ in $\mathscr{E}$ which has been reached in a partie at time t. A short time later—$t + h$—the play has progressed to the (variable) point $\mathbf{x}^0$. Then

$$\mathscr{P}(\mathbf{x}) = \text{payoff at } \mathbf{x} = \int_t^{t+h} G(\mathbf{x}, \phi, \psi)\, dt + V(\mathbf{x}^0)$$

for during the interlude $(t, t + h)$, the payoff acquires an increment equal to the above integral which, to get $\mathscr{P}(\mathbf{x})$, must be added to the payoff at $\mathbf{x}^0$ (which, of course, is the sum of such an integral extended over the remaining time of play and H at the terminal point). We assume that play from $\mathbf{x}^0$ is optimal, so that the payoff here is $V(\mathbf{x}^0)$. The idea of our reasoning is to regard $\mathbf{x}$ as fixed but to admit various $\mathbf{x}^0$, which arise from all possible choices of ϕ and ψ during $(t, t + h)$.

We shall replace the integral by a Taylor expansion of the type

$$\int_t^{t+h} f(x)\, dx = hf(t) + \tfrac{1}{2}h^2 f'(t + \theta h), \qquad 0 < \theta < 1.$$

In the final term we shall write

$$\mathbf{x}^0 = \mathbf{x} + \mathbf{u}$$

where, very closely for small enough h (because f_j is $\dot{x}_j$ for the ϕ and ψ chosen),

$$u_j = f_j(\mathbf{x}, \phi, \psi)h$$

and we shall use (4.1.1) to make a further replacement. When all these have been done, we find that

$$\mathscr{P}(\mathbf{x}) = V(\mathbf{x}) + h\left[G(\mathbf{x}, \phi, \psi) + \sum_j V_j f_j(\mathbf{x}, \phi, \psi) + \alpha(h)\right]$$

where the ϕ and ψ stand for their values at $\mathbf{x}$[5] and α goes to zero with h.

We are to take the minimax of $\mathscr{P}$ with respect to ϕ and ψ. Such means applying this operation to the bracket. But, as by definition the minimax of $\mathscr{P}$ is $V(\mathbf{x})$, the minimax of the bracket equals zero. Letting $h \to 0$, we have (4.2.1), the ME.

We shall refer hereafter to (4.2.1) as the first form of the main equation and designate it by ME_1. If we actually ascertain the minimax, the ϕ and ψ (or a judicious choice if there is more than one possibility) that supply it will in general depend on $\mathbf{x}\,(= x_1, \ldots, x_n)$ and the $V_i\,(i = 1, \ldots, n)$.

[5] Or better, some mean value during the interlude $(t, t + h)$.

It will be convenient to write the latter vector $\{V_1, \ldots, V_n\}$ as V_x.[6] Then the minimax in the ME_1 will be furnished by

$$\bar{\phi}(\mathbf{x}, V_x) \quad \text{and} \quad \bar{\psi}(\mathbf{x}, V_x). \tag{4.2.2}$$

When these functions are substituted into the ME_1, the bracket will be zero; we will have

$$\sum_j V_j f_j[\mathbf{x}, \bar{\phi}(\mathbf{x}, V_x), \bar{\psi}(\mathbf{x}, V_x)] + G[\mathbf{x}, \bar{\phi}(\mathbf{x}, V_x), \bar{\psi}(\mathbf{x}, V_x)] = 0 \tag{4.2.3}$$

a first-order partial differential equation in V which the Value must satisfy. We will refer to (4.2.3) as the second form of the main equation and denote it by ME_2.

We have already used $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ to denote optimal strategies. Note that (4.2.2), as they stand, are not strategies at all, for we are presuming that at this stage we do not know the V_i. However, in practice, this conflict of notation does not appear to create any confusion and seems preferable to some novel symbol, such as $\bar{\bar{\phi}}$, for (4.2.2). Note that once V is known and its partial derivatives inserted for the V_i in (4.2.2) these functions become the optimal strategies $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$, a fact which ameliorates the above possible confusion.

In the future, when an ME_2 is written, to save space the arguments $\mathbf{x}$, V_x of $\bar{\phi}$ and $\bar{\psi}$ will often be omitted. We shall take pains to label the equation ME_2, so that the reader may be aware of these unwritten symbols.

Exercise 4.2.1. The following KE will be those of Examples 4.4.1 to 4.4.5 to come (however, their comprehension is not necessary for this exercise). The payoff is terminal and the KE are

$$\dot{x} = u\psi + w \sin \phi$$
$$\dot{y} = -1 + w \cos \phi, \qquad -1 \leqslant \psi \leqslant 1$$

where u and w are smooth, positive functions of x and y. Write the ME_1 and show that the ME_2 is

$$uV_x\bar{\psi} - w\rho - V_y = 0$$

where, if $\rho = \sqrt{V_x^2 + V_y^2}$,

$$\bar{\psi} = \operatorname{sgn} V_x$$
$$\sin \bar{\phi} = -V_x/\rho, \quad \cos \bar{\phi} = -V_y/\rho.$$

(Use Lemma 2.8.1.)

[6] The customary notation being grad V. This notation is generally not required in particular problems. In such, if x denotes a state variable, V_x will of course mean $\partial V/\partial x$.

Exercise 4.2.2. The homicidal chauffeur game. Write the ME_1 and ME_2 for both the KE (Example 2.1.2) in the realistic space and the KE (Example 2.2.2) in the reduced space. One notation for the former will be $V_1, \ldots, V_5$, the subscripts pertaining to the state variables in the order in which they appear in the KE. (The reader will find the required ME in subsequent pages.)

Observe that along an optimal path

$$\dot{V} = \sum_j V_j f_j(\mathbf{x}, \bar{\phi}, \bar{\psi}) = -G(\mathbf{x}, \bar{\phi}, \bar{\psi})$$

and so V is constant on all such paths if and only if the payoff is terminal.

The second proof of the main equation depends on a new concept:

4.3. SEMIPERMEABLE SURFACES AND A SECOND DERIVATION OF THE MAIN EQUATION

We take it that each small portion of the surfaces under discussion separates the neighboring space. As orientation is germane to our purpose, we distinguish the two directions in which the surface may be penetrated, calling them the *P*- and *E-directions*. The "side" of the surface reached after penetration in the *P*-[*E*-] direction will be called the *P*-[*E*-] *side*. We take a point $\mathbf{x}$ on a so oriented surface and visualize the full vectogram at $\mathbf{x}$. We will say the surface is *semipermeable at* $\mathbf{x}$ when the following is true:

There is at least one value $\bar{\phi}$ of ϕ such that if $\phi = \bar{\phi}$,[7] no vector in ψ-vectogram penetrates the surface in the *E*-direction. Similarly, there is a $\bar{\psi}$[7] which prevents penetration in the *P*-direction.

A surface having this property at each point will be called a *semipermeable surface*, abbreviated SPS.

We have already seen (Theorem 2.4.1) that we can transform any game into one with terminal payoff. Consider an instance of the latter which we suppose solved and for which $V(\mathbf{x})$ has at least two values.

Any surface which separates the parts of $\mathscr{E}$ where $V > c$ and $V < c$ (c, any constant) must be semipermeable with V decreasing as the surface is crossed in the *P*-direction. For if, at some point $\mathbf{x}$ of the surface, there were no $\bar{\phi}$, *P* could not prevent *E* from pulling $\mathbf{x}$ into the side with the larger V. Similarly, there is a $\bar{\psi}$. Thus, for $\mathbf{x}$ on the surface, the players must employ $\bar{\phi}$ and $\bar{\psi}$, and these are the optimal strategies there.

[7] $\bar{\phi}[\bar{\psi}]$ will be used interchangeably to denote a value of $\phi[\psi]$ with the described property or the set of all such values. Even though we seem to have burdened the barred control variable with still a third meaning, it will soon coalesce with our former usages.

Now suppose in a certain subregion of $\mathscr{E}$, V is of class C_1 and constant in no neighborhood. Then the surfaces on which V is constant will be semipermeable. The vector $V_x = \{V_i\}$ is normal to such surfaces. Whether a moving point penetrates the surface in one direction or the other or not at all depends on the sign of its velocity component along this vector. That is, the semipermeability condition for the surfaces of constant V is

$$\min_{\phi} \max_{\psi} \sum_i V_i f_i(\mathbf{x}, \phi, \psi) = 0.^{8} \tag{4.3.1}$$

But this is the main equation in the terminal payoff case.

If we regard V_x as the flow velocity of some substance in $\mathscr{E}$, (4.3.1) can be interpreted as implying no flow across a SPS when $\bar{\phi}$ and $\bar{\psi}$ (yielding the min and max) are employed. Using but one of $\bar{\psi}$ and $\bar{\phi}$ and leaving the other free, we see that each player can prevent a transverse flow across the surface in his direction. Hence the name, semipermeable surface.

Suppose we had begun with a general game and used the transformation of Theorem 2.4.1 to obtain one with a terminal payoff. If there were n state variables originally, the sum in (4.3.1) will have $n + 1$ terms, and f_{n+1} will be G. We know that in the transformed game, H is (the old H) $+ s_n$, and the new optimal paths are translations of the old in the x_{n+1} direction. Thus if, at any starting point, x_{n+1} is increased, V will be increased by the same amount. Therefore $V_{n+1} = 1$.

The preceding two replacements show that the final term in the sum of (4.3.1) is G and so this equation is identical with the ME_1, (4.2.1).

The preceding conception furnishes an outlook on differential games with rather forceful intuitive appeal. For simplicity, let us envisage a case where everything turns out to be smooth; there are no singular surfaces.[9] The payoff is terminal; we know this can be made true of any game.

On $\mathscr{C}$, the function H is given. We assume the curves [$(n - 2)$-manifolds] on $\mathscr{C}$ of constant H cover this surface in a regular way. Now, suppose we succeed in filling $\mathscr{E}$ by a family of surfaces with just one passing through point of $\mathscr{E}$, such that

1. They are SPS, suitably oriented.
2. Each meets $\mathscr{C}$ in a curve of constant H.

(Our technique of solution, a basic constituent of this book, is logically equivalent to the construction of such a family.)

[8] A more detailed interpretation of this equation's being characteristic of smooth SPS will be found in Section 8.3.

[9] Such is generally true of the in the small constituents of more general cases.

Then it certainly seems reasonable to assert that these surfaces are of constant V (the constant being the value of H at $\mathscr{C}$). For at any point of $\mathscr{E}$, the minimizing player P, when suitably opposed, cannot cause $\mathbf{x}$ to penetrate the local surface to one of lower V. Nor can E, similarly, attain a higher V. In fact, to prevent his opponent from doing better than V (which labels the local surface) each is compelled to use the non-penetrating $\bar{\phi}$ and $\bar{\psi}$ which appeared in the definition of a SPS. As long as they continue to do so, $\mathbf{x}$ remains on the same surface; a defection by either player makes it possible for his opponent to penetrate to a more advantageous one.

Now, *if continued play with* $\bar{\phi}$ *and* $\bar{\psi}$ *will bring* $\mathbf{x}$ *to* $\mathscr{C}$, we feel a strong assurance that the labels on the surfaces are values of the Value. The italicized assumption above is typical of the domain of games of kind. We will broach the subject in this chapter with two typical illustrations (Examples 4.4.4 and 4.4.5).

The theorem of the following section is an adaptation of the foregoing ideas to K-strategies.

4.4. THE VERIFICATION THEOREM

Inasmuch as the diversity of phenomena that will soon be seen to arise in even the most typical problems is great enough to preclude an adequate existence theorem, we shall adopt another approach. We shall develop a technique for solving problems. The question will then be whether the formal solution so obtained—the Value $V(\mathbf{x})$ and the accompanying optimal strategies $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$—really solves the problem, and if so, in what sense.

The sense will be that of K-strategies. The means is Theorem 4.4.1. It may be applied separately to the stages of the solution process, each of which is essentially the finding of an integral of the main equation which satisfies the proper boundary conditions, such as agreeing with H on $\mathscr{C}$.

We shall state and prove Theorem 4.4.1 and demonstrate its use with some examples. Let us recall our general assumptions that each $f_j(\mathbf{x}, \phi, \psi)$ is a continuous function of its three arguments and the constraints limit ϕ and ψ to sets; we shall call these sets E_ϕ and E_ψ, and they are compact and independent of $\mathbf{x}$. First the

LEMMA 4.4.1. If ϕ and ψ are held constant and $\mathbf{x}$ is confined to a compact set E_x, then $f(\mathbf{x}, \phi, \psi)$ is uniformly continuous with respect to $\mathbf{x}$ and all possible constant values of ϕ and ψ.

Proof. We have to show that for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon)$ such that

$$|f(\mathbf{x}^1, \phi, \psi) - f(\mathbf{x}^2, \phi, \psi)| < \varepsilon$$

for all $\mathbf{x}^1, \mathbf{x}^2 \in E_x$ with

$$|\mathbf{x}^1 - \mathbf{x}^2| < \delta$$

and all $\phi \in E_\phi$ and $\psi \in E_\psi$. But f is a continuous function of all three (vector) arguments in the compact product set $E_x \times E_\phi \times E_\psi$ and hence uniformly continuous therein. The lemma is a special case of this uniform continuity.

THEOREM 4.4.1. Let $\mathscr{E}'$ be a subregion of $\mathscr{E}$ neighboring a subregion $\mathscr{C}'$ of $\mathscr{C}$ in a game of degree with terminal payoff. If a function $V(\mathbf{x})$, defined in $\mathscr{E}'$, enjoys the properties:

1. It satisfies the ME$_1$ (4.3.1) in $\mathscr{E}'$
2. It is of class $\mathscr{C}_1$ in $\mathscr{E}'$
3. It equals H on $\mathscr{C}'$
4. It is the only function satisfying 1, 2, and 3.

If $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ are any functions which furnish the minimax in (4.3.1), then $V(\mathbf{x})$ is the Value in $\mathscr{E}'$ in the sense of K-strategies and $\bar{\phi}$ and $\bar{\psi}$ are optimal tactics, provided that $\mathbf{x}$ reaches $\mathscr{C}'$ from any starting point in $\mathscr{E}'$.

Proof. In Section 2.7 (end), we learned that, by a change in the time scale, we can bound all speeds in each full vectogram in $\mathscr{E}'$ without changing the game in any essential way. Let us do so with the bound $\leqslant 1$.

Let us select a tactic $\bar{\phi}(\mathbf{x})$ for P such that for each $\mathbf{x}$, $\bar{\phi}$ is minimizing in (4.3.1). Let E play any K-strategy; let $\psi(\mathbf{x})$ be its tactic. Play starts from $\mathbf{x}^0$. Given an $\varepsilon > 0$, we are going to complete P's K-strategy by constructing a σ_t. We shall speak as if P were to play indefinitely; of course, we need but curtail our scheme when $\mathscr{C}$ is reached.

First we divide time into unit intervals I_n: $n \leqslant t < n + 1$ ($n = 0, 1, 2, \ldots$). During I_n, $\mathbf{x}$ cannot have traveled more than $n + 1$ from the starting position $\mathbf{x}^0$ and so is constrained to a compact set. As each V_i is continuous by 2 and from the lemma, the function $\Sigma_i V_i(\mathbf{x}) f_i(\mathbf{x}, \phi, \psi)$ is uniformly continuous in this compact set when ϕ and ψ are constants and uniformly with regard to these constants too.

Pick m_n so that if $\mathbf{x}$ changes in distance by less than $1/m_n$, the quantity $\Sigma_j V_j f_j$ will change by less than $\varepsilon/2^{n+1}$. Next subdivide I_n into m_n equal parts, completing σ_t. Then, during the passage of any of these subinterludes, the change in $\Sigma_j V_j f_j$ suffers the same bound as above. A sub-interlude may again be subdivided by one or more t_k' from E's K-strategy. But at the beginning of each of length $1/m_n$, because P then used a minimizing ϕ in (4.3.1), we have

$$\sum_j V_j f_j(\mathbf{x}, \bar{\phi}, \psi) \leqslant 0.$$

The arguments ϕ, ψ being the piecewise constant tactical values from the

K-strategies played, the left side cannot rise to $\varepsilon/2^{n+1}$ during any such interval and therefore cannot throughout I_n.

Now let us consider the path of $\mathbf{x}$ under the preceding K-strategies. In each of the ultimate subinterludes, when both ϕ and ψ are constant, it is an integral of the KE with these constant arguments; thus the path is a "polygonal" sequence of smooth segments. On the path as a whole V is continuous, and on each segment dV/dt exists and is given by $\Sigma_j V_j f_j(\mathbf{x}, \phi, \psi)$ with appropriately constant ϕ and ψ. From the preceding paragraphs it follows that the growth of V is less than $\varepsilon/2^{n+1}$ during I_n and hence always less than $\varepsilon(\frac{1}{2} + \frac{1}{4} + \ldots) = \varepsilon$.

By definition and from 3, the payoff will be the $V(\mathbf{x})$ when $\mathbf{x}$ reaches $\mathscr{C}'$. Therefore

$$\text{payoff} < V(\mathbf{x}^0) + \varepsilon.$$

Similarly, we can construct a $\sigma_t{}'$ for E ensuring a payoff $> V(\mathbf{x}^0) - \varepsilon$. Thus V is the value.

The application of the verification theorem may be multiple; such is accomplished according to the following general idea: Let $\mathscr{E}'$ and $\mathscr{C}'$ be as in the hypothesis of the theorem. Pass a second surface $\mathscr{C}''$ through $\mathscr{E}'$. If we use this surface as we did $\mathscr{C}'$, taking for H on $\mathscr{C}''$ the values of $V(\mathbf{x})$ obtained from the first solution, the new solution on the side of $\mathscr{C}''$ away from $\mathscr{C}'$, will agree with the old. This principle permits the validation of a complex solution in consecutive stages. Singular surfaces of various types can be handled as $\mathscr{C}''$ above, referring now to a new phase of the solution rather than confirming an old one.

What of the very embracing final clause of the theorem? Whether or not termination will occur is a matter of games of kind and their treatment too falls under the aegis of K-strategies. The subject is treated in detail in Chapter 8, but we must make some anticipations here.

Since we are dealing with games, we shall suppose one player, say P, desires termination while his opponent does not. In the most interesting cases (and the only ones discussed in this book) both termination and nontermination starting points exist. The former are such that P has a strategy ensuring termination against any opposition; the latter are such that E can similarly prevent it. The two sets of points are separated by a surface, which must be semipermeable, called the barrier.[10] Let us imagine the barrier imbedded in a lamina consisting of neighboring "parallel" SPS and define (temporarily) a smooth function $U(\mathbf{x})$ on the lamina which equals zero on the barrier, is strictly decreasing in the termination or P-direction, and constant on each SPS. Then U satisfies the same equation (4.3.1) as V and so we can construct a K-strategy for P

[10] These ideas are amplified in Chapter 8.

as in the preceding proof. For any starting point $\mathbf{x}^0$ on the termination side, so that $U(\mathbf{x}^0) < 0$, if we use an $\varepsilon \leqslant -\frac{1}{2}U(\mathbf{x}^0)$, then for all $\mathbf{x}$ of the ensuing path, $U(\mathbf{x}) < 0$. Then, no matter what K-strategy E plays, he cannot make $\mathbf{x}$ cross the barrier into the nontermination points. Similarly, for starting points on the other side, E will have a K-strategy deterring entry into the termination region.

Once the question of termination or not is settled in this or a similar way, we can apply Theorem 4.4.1 with $\mathscr{E}'$ consisting only of the termination points. We now have the requisite assurance that termination will occur.

Throughout this book we shall solve many examples in the sense of ascertaining entities $V(\mathbf{x})$, $\bar{\phi}(\mathbf{x})$, $\bar{\psi}(\mathbf{x})$ as discussed in Section 4.1. In principle we should apply the verification theorem to each to prove that the formal solutions are actually such. But as this would lead to a pattern of monotonous replication, we shall give here a series of simple yet typical illustrations—all variations of the same example—which will clarify many aspects of the general procedure.

This chain of examples will unavoidably necessitate our anticipation of certain later ideas. The reader can either read these cases lightly now and return to them after subsequent ingestion, or he can do some (light) reading in the requisite advanced pages.

We pause first to recall the germ of the K-strategy idea. When we have found, say, $\bar{\phi}(\mathbf{x})$ and employ it as an optimal tactic, it can be pitted against *any* $\psi(\mathbf{x})$ which fulfills the constraints. We are freed from putting any constrictions of function type, such as piecewise continuity, differentiability, etc., on the opponent's strategy.

Example 4.4.1. $\mathscr{E}$ is the upper half-plane and $\mathscr{C}$ is the x-axis. The vectogram for E is shown at Figure 4.4.1a; it has a downward unit vertical component and a horizontal headline[11] of half-length $u(x, y)$, a positive and smooth function. That of P is circular of radius $w(x, y)$, again a smooth, positive function with always $w < u$ and, for some constant c, $w \leqslant c < 1$. The total velocity for x is to be the vector sum of a choice from each vectogram. Analytically all this means that the kinematic equations are

$$\dot{x} = u(x, y)\psi + w(x, y) \sin \phi$$

$$\dot{y} = -1 + w(x, y) \cos \phi, \qquad -1 \leqslant \psi \leqslant 1.$$

The payoff will be terminal with $H = x$ on $\mathscr{C}$ (where $y = 0$). Thus E will strive to have $\mathbf{x}$ reach $\mathscr{C}$ at a point as far right as possible, and P similarly struggles for the left.

Always E will play his rightmost vector ($\psi = 1$ in the KE). At (b) of the

[11] That is, the line of "arrowheads" of the vectors.

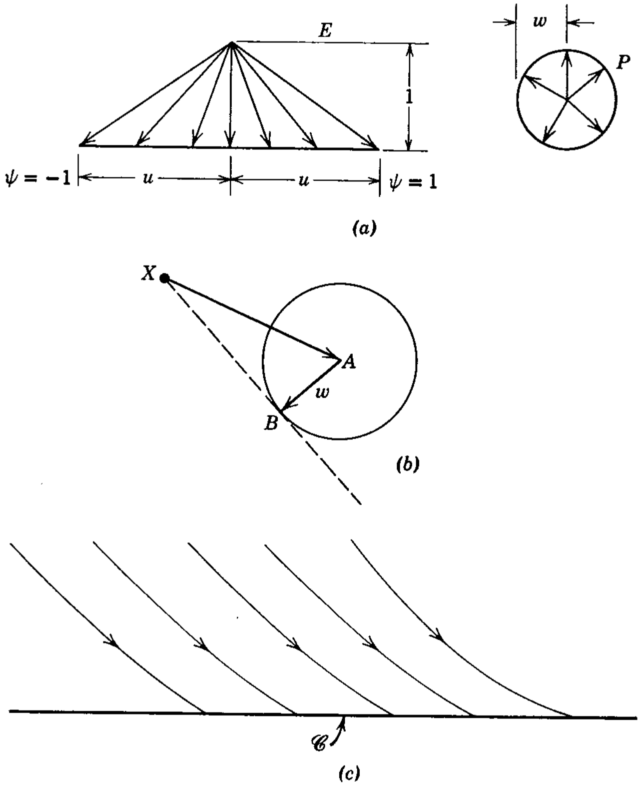


Figure 4.4.1

figure let XA be this vector. The dashed line XB is tangent from X to a circle of radius w (w is reckoned at X) and center A. Then XB is a properly oriented semipermeable direction. If a family of curves is drawn (an ordinary differential equation solved) having these directions as that of their tangents at each point, these curves will be SPS. If each is labeled with the value of H at its meeting with $\mathscr{C}$, the labelings will constitute $V(\mathbf{x})$.

These assertions either follow from our analytic methods still to come or they can be attained geometrically. To see that XB is semipermeable, consult Lemma 10.2.2, which deals with a very similar situation.

The curves of constant V, which are also the optimal paths, perhaps appear as in (c) of the figure. The four hypotheses of Theorem 4.4.1 are all satisfied by the above $V(\mathbf{x})$. Further, as the downward velocity of $\mathbf{x}$ is always at least 1-c, we are assured that every partie will terminate. Thus there is no difficulty in asserting the theorem's conclusion.

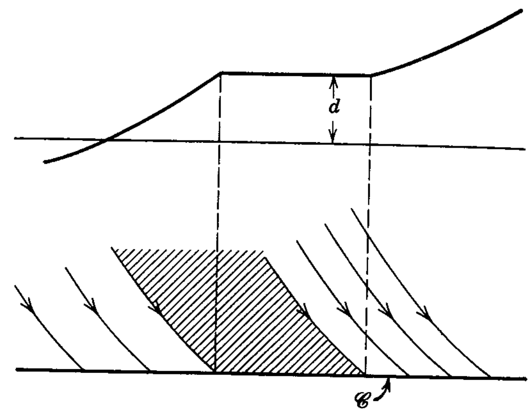


Figure 4.4.2

Example 4.4.2. The same, except we change H. It is increasing except on a certain interval where it is constant, say d; its graph is as in Figure 4.4.2a. Such causes a region (shaded in (b) of the figure) in $\mathscr{E}$ where V is constant; in fact $V = d$.

Because here all $V_i = 0$, the main equation is certainly satisfied. As any ϕ and ψ give (trivially) the minimax, all are optimal. Considering the nature of the game, this is certainly true in the shaded region. The theorem holds.

The point here is simply that, whereas V is unique, $\bar{\phi}$ and $\bar{\psi}$ do not have to be so.

Example 4.4.3. Again the same, except that now $H = x^2$. Thus E will strive to have $\mathbf{x}$ cross $\mathscr{C}$ far from, and P near to, the point where $x = 0$.

It is evident that at points far to the right in $\mathscr{E}$, our previous construction holds. For points far to the left, a symmetric one holds with E playing his leftmost vector, etc.

Thus we construct two families of paths as shown in Figure 4.4.3. Chapter 6 deals with such cases; there we learn that we should delete the paths after they cross (in the sense of traversing them *from* $\mathscr{C}$) a curve $\mathscr{D}$ which is the point set where the two functions $V(\mathbf{x})$ are equal ($\mathscr{D}$ will later be termed a dispersal surface).

For $\mathbf{x}$ on either side of $\mathscr{D}$ we apply the theorem, but we must take $\mathscr{E}'$ as that part of $\mathscr{E}$ lying on one side of $\mathscr{D}$ and $\mathscr{C}'$ the part of $\mathscr{C}$ lying on the same side of O and $\mathscr{D}$ itself. For either part H is taken as given on the retained part of $\mathscr{C}$ and as the mutual value of V on $\mathscr{D}$. In this way all hypotheses of the theorem are fulfilled and we can appropriate its conclusion. Observe that no other dividing curve but $\mathscr{D}$ can accomplish this result.

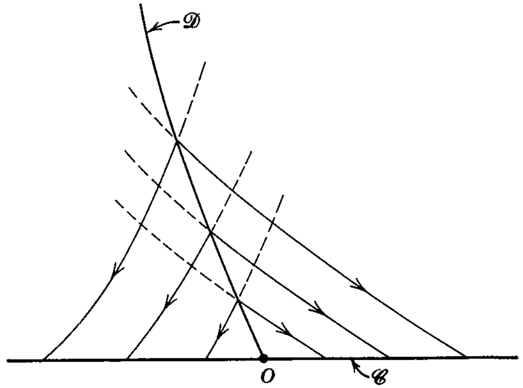


Figure 4.4.3

For starting points on $\mathscr{D}$ each player is faced with two choices of values of his control variables. Each should play a mixed strategy making his choice with probabilities $(\frac{1}{2}, \frac{1}{2})$. This matter is discussed at length in Chapter 6.

There are two points to note here. We curtailed part of the solution of the main equation: therefore, not all of every formal solution $V(\mathbf{x})$ need be the Value of the game. A more subtle instance occurs in connection with equivocal surfaces and will be expounded in detail in Chapter 10.

The second point is that $\mathscr{D}$ is a singular surface; although $V(\mathbf{x})$ is continuous, certainly all its first partials do not exist there. The theorem is quite capable of dealing with problems that have singular surfaces when judiciously applied.

Example 4.4.4 We now let $\mathscr{E}$ be the entire plane and $\mathscr{C}$ the positive x-axis. H can be taken as x as in Example 4.4.1.

What is new is that from starting points far to the left $\mathbf{x}$ may miss $\mathscr{C}$ and there will be no termination. It is more interesting to have P desire termination and E, its avoidance. That is, E wishes to escape (nontermination) if possible; if not, he wishes $\mathbf{x}$ to meet $\mathscr{C}$ as far to the right as he can occasion.

We could realize the foregoing situation numerically by assigning the payoff $+\infty$ in the event of escape, but such seems a rather empty formalism.

We have explained very briefly, on page 74 the nature of a barrier and referred the reader to Chapter 8 for fuller details. In our case the barrier will be the "left-hand" semipermeable surface, in the sense of Example 4.4.3, which passes through O. It is marked $\mathscr{B}$ in Figure 4.4.4.

As was explained earlier, if $\mathbf{x}$ lies to the left of $\mathscr{B}$, E has a K-strategy which guarantees $\mathbf{x}$'s not crossing $\mathscr{B}$ and hence ensures escape. Similarly, if $\mathbf{x}$ lies to the right of $\mathscr{B}$, P has a K-strategy which deters crossing (in the opposite sense) and so ensures termination. (We make no statement now concerning the case of $\mathbf{x}$'s starting from $\mathscr{B}$.)

For starting points to the right of $\mathscr{B}$ and above $\mathscr{C}$, both players, knowing that capture is inevitable, play the game of degree. The reasoning is just that of Example 4.4.1 except that $\mathscr{E}'$ is taken as the above subset of $\mathscr{E}$ bounded by $\mathscr{B}$ and $\mathscr{C}$.

The fact that the optimal strategies of P—that of assuring termination and that of minimizing the numerical payoff—are distinct has little effect. He never need be concerned with the former unless $\mathbf{x}$ gets very close to $\mathscr{B}$. (As there is no lower limit to this closeness, we might say the latter strategy is optimal throughout the interior of $\mathscr{E}'$.) But should E be foolish enough to make some attempt at escaping, P can wait to frustrate him until $\mathbf{x}$ gets very close (if it does) to $\mathscr{B}$, meanwhile playing the game of degree optimal strategy; E will not escape but will be penalized in payoff also.

Again we have part of the formal solution $V(\mathbf{x})$ (to the right of $\mathscr{B}$) not being the Value. But the main point here relates to the final clause of the theorem. We sketch another such instance.

Example 4.4.5. The same as Example 4.4.1 except for the inequalities on w. Far to the right in $\mathscr{E}$, $w(x, y)$ is subject to the earlier strictures, but as x decreases, w grows until at the far left $w > 1$ and $w > u$. From such starting positions P has command of the motion of $\mathbf{x}$; he can avoid termination if he wishes.

Let it be that he does so wish and E seeks it when possible. Such is the case sufficiently rightward in $\mathscr{E}$.

Again it is possible to segregate the termination and escape points by an SPS of the type which in Chapter 8 will be called a natural barrier. Any premature details here would fill an undue amount of space.

Once this barrier is known, the discussion becomes very much like that of the last example and will not be repeated.

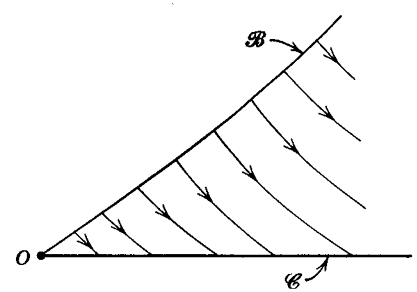


Figure 4.4.4

The only point we wished to make here is possibly superfluous. In all the previous examples the question ranked strong of the inevitability of termination. Only in 4.4.4 did we admit the alternative possibility. Did our means seem artificial there? First, assurance by bounding the downward velocity of **x** from below; then negation, by cutting off $\mathscr{C}$. We wished to show here that our ideas stand up under "natural" circumstances too.

4.5. THE PATH EQUATIONS

We now revert to the technique for obtaining solutions. The contents of this and succeeding sections will be used extensively throughout the rest of this book.

Our starting point will be the ME_2, which we rewrite for reference:

$$\sum_i V_i f_i(\mathbf{x}, \bar{\phi}, \bar{\psi}) + G(\mathbf{x}, \bar{\phi}, \bar{\psi}) = 0 \tag{4.2.3}$$

where $\bar{\phi}$ and $\bar{\psi}$ are functions (4.2.2) of the vectors in **x** and V_x.

We differentiate (4.2.3) with respect to each x_k. Doing so according to the rules of elementary calculus, we examine the components as they arise. First we have

$$\sum_i \frac{\partial V_i}{\partial x_k} f_i. \tag{4.5.1}$$

But since
$$\frac{\partial V_i}{\partial x_k} = \frac{\partial^2 V}{\partial x_i \, \partial x_k} = \frac{\partial V_k}{\partial x_i} \text{ and } f_i = \dot{x}_i,$$

$$(4.5.1) = \sum_i \frac{\partial V_k}{\partial x_i} \dot{x}_i = \dot{V}_k.$$

The latter term would be the time derivative of V along the resulting path were $\bar{\phi}$ and $\bar{\psi}$ played. Next we have

$$\sum_i V_i f_{ik} + G_k$$

where $f_{ik}(\mathbf{x}, u, v) = \dfrac{\partial f_i}{\partial x_k}(\mathbf{x}, u, v)$ and $G_k = \dfrac{\partial G}{\partial x_k}$. Then we encounter

$$\sum_{j=1}^{\lambda} \frac{\partial}{\partial \phi_j}\Big(\sum_i V_i f_i + G\Big)\frac{\partial \bar{\phi}_j}{\partial x_k}. \tag{4.5.2}$$

Each ϕ_j is supposed subject to constant bounds as discussed in Section 2.7. The minimizing $\bar{\phi}_j$ occurs either interior to the constraining interval or at an endpoint. If the former, the $\partial/\partial\phi_j(\cdot)$ term of (4.5.2) is 0 because of the minimizing property of $\bar{\phi}$; if the latter, the $\partial\bar{\phi}_j/\partial x_k$ is 0 as $\bar{\phi}_j$ is constant.

In either case then (4.5.2) vanishes. The same is true of the remaining terms devolving on the $\bar{\psi}_i$. We conclude

$$\dot{V}_k = -\Big\{\sum_i V_i f_{ik}[\mathbf{x}, \bar{\phi}(\mathbf{x}, V_x), \bar{\psi}(\mathbf{x}, V_x)] + G_k[\mathbf{x}, \bar{\phi}(\mathbf{x}, V_x), \bar{\psi}(\mathbf{x}, V_x)]\Big\}. \tag{4.5.3}$$

Using the barred control variables in the KE themselves gives further

$$\dot{x}_k = f_k[\mathbf{x}, \bar{\phi}(\mathbf{x}, V_x), \bar{\psi}(\mathbf{x}, V_x)]. \tag{4.5.4}$$

This set (4.5.3), (4.5.4) of $2n$ ordinary differential equations in the $2n$ unknowns x_k, V_k shall be called the *path equations*. Actually they are the characteristic equations[12] of (4.2.3) (slightly special in that the terms (4.5.2) are nullified). Solutions of the ME_2 can be built from integrals of the path equations in the standard manner,[12] a procedure we shall shortly adapt to our purposes.

Note that the brace in (4.5.3) is nothing more than the formal derivative of (4.2.3) with respect to x_k, that is, we ignore all but the explicit appearances of x_k in (4.2.3).

4.6. THE RETROGRESSION PRINCIPLE

When solving a game we reverse time; we start at $\mathscr{C}$ and work backward into $\mathscr{E}$. From the point of view of differential equations the motive is clear. In the previous section we effectively reduced the construction problem to the integration of the path equations. We cannot obtain the appropriate particular integrals without initial conditions. And at the outset of a problem we will have the needed data only on $\mathscr{C}$ where we know $V = H$.

But perhaps a stronger, if more heuristic, insight can be gleaned from the discrete examples of Chapter 3, where building backward from the terminal states was plainly the only means at our disposal. And it would appear that any differential game can be quantized into a discrete one.

We shall henceforth use the symbol τ for the time needed for **x** to reach $\mathscr{C}$ (or some other surface playing a like role) so that on any optimal path

$$\tau = \text{constant} - t.$$

The symbol $\mathring{x}$ will mean $\partial x/\partial\tau$ so that

$$\mathring{x} = -\dot{x}.$$

The path equations, when rewritten in this retrogressive notation (the

[12] See, for example, Reference [6].

signs of the right sides are changed), will be referred to as the RPE (Retrogressive Path Equations) and will be an essential constituent of our standard procedure:

$$\overset{\circ}{x}_k = -f_k(\mathbf{x}, \bar{\phi}, \bar{\psi}) \tag{4.6.1}$$

$$\overset{\circ}{V}_k = \sum_i V_i f_{ik}(\mathbf{x}, \bar{\phi}, \bar{\psi}) + G_k(\mathbf{x}, \bar{\phi}, \bar{\psi}). \tag{4.6.2}$$

Again we note that the right side of (4.6.2) is the derivative of (4.2.3) with respect to explicitly appearing x_k.

Let us recall that (4.6.2) was derived on the assumption of constant constraints for the control variables. Although such can always be brought about, occasionally problems arise where it is inconvenient to do so.[13] In these cases we must adjoin terms to (4.6.2) corresponding to suitable derivatives of the proper control variables.

It is interesting to note that the RPE are Hamilton-Jacobi equations, although we have found no means as yet of capitalizing on this fact.

Exercise 4.6.1. Find the RPE for the ME_2 given in Exercise 4.2.1. Take $u = 1 + x^2$, $w = \frac{1}{2}e^{-3v}$. Verify that the left side of the ME_2 has zero derivative with respect to τ.

Exercise 4.6.2. Find the RPE for the homicidal chauffeur game in the realistic space. The ME_2 is here

$$w_1(V_1 \sin\theta + V_2 \cos\theta) + w_2\rho + \frac{w_1}{R} V_5\bar{\phi} + 1 = 0$$

where
$$\rho = \sqrt{V_3^2 + V_4^2}, \quad \bar{\phi} = -\text{sgn}\, V_5$$

and
$$\sin\bar{\psi} = \frac{V_3}{\rho}, \quad \cos\bar{\psi} = \frac{V_4}{\rho}.$$

The KE are on page 28.

(The answers to these two exercises appear at the end of this chapter.)

Problem 4.6.1. From the solution to the last exercise, show that for the solution in the small of the homicidal chauffeur game, P always turns sharpest possible right or left and E travels in a straight line. Thus we can expect the solution to have many singular surfaces.

4.7. THE INITIAL CONDITIONS

We use this term in the retrogressive sense. We are concerned with known values of $\mathbf{x}$ and V_x on $\mathscr{C}$ (or some other surface playing a like role) which can serve as initial conditions with respect to τ when we integrate

[13] For example, Example 5.6.

the RPE. But in the progressive sense, that of t, the way a partie actually develops, they are final conditions.

In many games not all of $\mathscr{C}$ is capable of serving as a receptor for termination. In the homicidal chauffeur game, for example, if we regard $\mathscr{C}$ as the planform of the car driven by P, it is clear that only the front bumper will be effective; it is rather difficult, when driving forward, to run over a fleeing pedestrian with the rear of one's car.

To study this phenomenon in general consider a position very near $\mathscr{C}$. One or the other player may be able to force or deter an imminent termination despite any opposition from his opponent. Let $\nu = (\nu_1, \ldots, \nu_n)$ be the vector normal to $\mathscr{C}$ from point $\mathbf{x}$ on $\mathscr{C}$ and extending into $\mathscr{E}$. If

$$\min_{\phi} \max_{\psi} \sum_i \nu_i f_i(\mathbf{x}, \phi, \psi) > 0 \tag{4.7.1}$$

then E can prevent immediate termination from a position sufficiently near $\mathbf{x}$. If (4.7.1) holds with the inequality reversed, P can compel immediate termination.

There is the question of whether a player will benefit from the exercise of such power. Sometimes the answer is obvious. We cite the case of termination time payoff; clearly E will defer termination whenever he can. But in other instances E may see that avoidance of the frying pan now will only lead to the fire later. We leave the intricacies of such questions to individual cases. However, in all yet encountered there have been no such difficulties. For definiteness, let P desire termination and let it be to E's advantage to avoid it if he can. Then we have found that whenever (4.7.1) obtains, E will defer an imminent ending.

Similarly, at those points of $\mathscr{C}$ where (4.7.1) holds with the inequality reversed, P will occasion termination immediately. These points will be called the *useable part* of $\mathscr{C}$. Termination will occur only at the useable part under optimal play.

The subset of $\mathscr{C}$, where (4.7.1) holds as it stands is the *nonuseable part.*

The curve [$(n - 2)$-manifold] on $\mathscr{C}$ which separates these parts, that is, for which

$$\min_{\phi} \max_{\psi} \sum_i \nu_i f_i(\mathbf{x}, \phi, \psi) = 0 \tag{4.7.2}$$

is called the *boundary of the useable part*, abbreviated BUP.

Of course, in many problems all of $\mathscr{C}$ is the useable part, for instance, those which include a kinematic equation, $\dot{T} = -1$, and $\mathscr{C}$ lies in the plane where $T = 0$. But in many others, ascertaining the useable part is an early and important step in the solution.

The initial conditions we need for integrating the RPE are then the values of x_i and V_i ($i = 1, \ldots, n$) on the useable part. The parametric representation of $\mathscr{C}$,

$$x_i = h_i(s_1, \ldots, s_{n-1}) \tag{4.7.3}$$

when restricted thereto, gives us the first n. To obtain the V_i on the useable part we recall that on $\mathscr{C}$, $V = H = H(s_1, \ldots, s_{n-1})$. Differentiating with respect to s_k gives us

$$\frac{\partial H}{\partial s_k} = \sum_i V_i \frac{\partial h_i}{\partial s_k}, \qquad k = 1, \ldots, n-1 \tag{4.7.4}$$

a set of $n - 1$ equations for the n unknowns V_i. The remaining equation is the ME_2 with x_i replaced by (4.7.3).

Sometimes a double solution to the set will appear. The reason is that there is nothing in our analysis to distinguish the two sides of $\mathscr{C}$. For example, in a pursuit game, $\mathscr{C}$ will often bound a region corresponding to capture. We are interested only in **x**'s encountering $\mathscr{C}$ so as to pass from its exterior to this interior capture region. But a game with the reversed crossing of $\mathscr{C}$ as termination also has a valid interpretation.[14]

In working problems there is always some simple way of telling which of the two possible solutions to retain. As a workaday matter, we warn the novice to be careful; here is an easy point on which to err.

Exercise 4.7.1. Obtain the useable part for the homicidal chauffeur game, using the reduced space. For $\mathscr{C}$ take the circle of radius l:

$$x = l \sin s, \qquad y = l \cos s.$$

Find the initial conditions on the useable part both for paths which emanate from $\mathscr{C}$ to its exterior and interior.

Our technique for solving problems in the small is now virtually complete. After integrating the RPE with the initial conditions just explained we obtain the $2n$ functions x_i and V_i of the n arguments

$$\tau, s_1, \ldots, s_{n-1}. \tag{4.7.5}$$

We should now invert the first n functions and solve for (4.7.5) as functions of the x_i. (This is at times a formidable formal elimination, but in most particular cases we can get around such difficulties with a bit of ingenuity.) We can then find V either by inserting the newly found functions in the second n integrals, thus obtaining $V_i(x_1, \ldots, x_n)$ and then integrating these to get V to within an additive constant, which is fixed by V's known value on $\mathscr{C}$, or we can work directly with $\int G\,dt + H$. The optimal strategies are known by inserting the x_i and V_i, after the elimination, into the $\bar{\phi}(\mathbf{x}, V_x)$ and $\bar{\psi}(\mathbf{x}, V_x)$, which accompanied the ME_2.

Of course, the solution in the large entails the study of singular surfaces, and many subsequent chapters will be devoted to particular types.

[14] If the interior of $\mathscr{C}$ was the set of points under surveillance by a detection device, such as radar, carried by P and E's objective is to slip out of range and P's to prevent him.

Answers

Exercise 4.6.1.

$$\mathring{x} = -(1 + x^2)\bar{\psi} + \tfrac{1}{2}e^{-3y}\frac{V_x}{\rho}, \qquad \mathring{V}_x = 2xV_x$$

$$\mathring{y} = 1 + \tfrac{1}{2}e^{-3y}\frac{V_y}{\rho} \qquad \mathring{V}_y = -\tfrac{3}{2}e^{-3y}\rho$$

where $\bar{\psi} = \text{sgn } V_x$, $\quad \rho = \sqrt{V_x^2 + V_y^2}$.

Exercise 4.6.2.

$$\mathring{x}_1 = -w_1 \sin\theta \qquad \mathring{V}_1 = 0$$

$$\mathring{y}_1 = -w_1 \cos\theta \qquad \mathring{V}_2 = 0$$

$$\mathring{x}_2 = -w_2\frac{V_3}{\rho} \qquad \mathring{V}_3 = 0$$

$$\mathring{x}_3 = -w_2\frac{V_4}{\rho} \qquad \mathring{V}_4 = 0$$

$$\mathring{\theta} = \frac{-w_1}{R}\bar{\phi} \qquad \mathring{V}_5 = w_1(V_1\cos\theta - V_2\sin\theta)$$

$$(\bar{\phi} = -\text{sgn } V_5, \quad \rho = \sqrt{V_3^2 + V_4^2}).$$

CHAPTER 5

Mainly Examples; Transition Surfaces; Integral Constraints

We shall put the theory developed thus far to work by solving several examples.[1] Essentially the only type of irregular behavior occurs on what are termed transition surfaces. The section immediately following gives an account of these singular surfaces adequate for the ensuing problems.

Our opening example parallels the debut of the calculus of variations. In 1696 John Bernoulli challenged the mathematical world with his problem of the brachistochrone or curve of quickest descent. He, Newton, Leibniz, L'Hospital, and Euler all reached solutions. The variational calculus was born.

A weight, subject to frictionless, constant Newtonian gravity, descends from a given starting point to a second lower one. If it is constrained to some curved path, which one minimizes the time of descent? The destination may be a curve instead of a point.

Instead of viewing the falling object as constrained to a scheduled route, we may think of it as being steered at each instant by a pilot who can choose the travel direction freely. For Newtonian mechanics governs only the speed—the familiar $v = \sqrt{2gh}$—and exercises no other jurisdiction on the motion. Thus the pilot may always choose from a circular vectogram.

There is no reason why we cannot imagine a second player, with a second superposed vectogram, whose object is to maximize the time of descent. We do so and are in the realm of game theory with the problem of the dolichobrachistochrone, the curve of slowest-quickest descent.

Section 5.4, following a study of this problem, contains a generic

[1] This chapter is a re-edited version of Rand Report RM-1486 (25 March 1955), entitled *Mainly Examples,* which does not include the section on integral constraints.

answer to the question provoked. What is the relation between our subject and classical calculus of variations?

The second problem is a simplified, primary version of a major question of military policy. In a protracted war, how should a combatant split his efforts between the long-range objective of attrition—destroying the enemy's sources of weapon supply—and the more immediate one of directly attacking military targets. Simple as is our model, the answer is not an obvious one, quantitatively at least. In Chapter 11 we will resume this problem with a discussion of its relation to reality, and Section 11.9 will contain a modified version which has a strikingly different solution.

The isotropic rocket problem of Section 5.5 is a pursuit game in which the pursuer, who steers by control of the direction of his driving thrust, chases an evader with simple motion. Like the homicidal chauffeur model, there is the possibility of a "swerve" under certain conditions, but unlike it, the solution is generally smooth and analytic rather than segregated by a multitude of singular surfaces. The solution appears to be a mélange of the straightforward and the unobvious.

The final problem is an application of our techniques to economic programming. One-player games can be treated by thinking of the opponent as inactive; his range of decision is null. What remains is a problem of direct optimization. The present instance is an idealized picture of steel production in which the current supply is allocated between use as an ingredient in making more steel, building more steel mills, and the stockpile. What such allocation at each instant maximizes the steel supply at the termination of the program?

The chapter ends with a technique for problems where one player at least is subject to a constraint which can be expressed as an obligation to keep constant an integral over the path of a function of $\mathbf{x}$, ϕ, and ψ. Such is the counterpart here of a classical side condition in the calculus of variations. Lack of space limits us here to but some new versions of a famous problem, but the idea is used more fruitfully in the bomber and battery game in the Appendix.

5.1. TRANSITION SURFACES

This, possibly the simplest and most direct, type of singular surface can arise only in conjuction with linear vectograms. Let one of the control variables, say ϕ_1, appear lineally in each KE and G (as defined in Section 2.4) with coefficients that are independent of the others. Then, when we construct the ME, ϕ_1 will appear lineally there too; its coefficient A will involve at most the x_i and V_i.

Let a and b be the constant constraints on ϕ_1:

$$a \leqslant \phi_1 \leqslant b.$$

Suppose the solution were known at a point $\mathbf{x}$ of $\mathscr{E}$. Then A is known there, and so supposing ϕ_1 to be minimizing, we would have at $\mathbf{x}$

$$\bar{\phi}_1 = b \text{ if } A < 0$$
$$\bar{\phi}_1 = a \text{ if } A > 0.$$

For definiteness, let us suppose the former holds throughout some neighborhood of $\mathbf{x}$. Along the optimal paths through these points, $\bar{\phi}_1 = b$ as long as $A < 0$. But suppose there occurs, as we follow each path (increasing τ), a point where first $A = 0$. Normalcy implies that these points constitute a surface $\mathscr{T}$.

To ascertain the solution beyond $\mathscr{T}$ we can use this surface as a seat of initial conditions, that is, it plays the role of $\mathscr{C}$, and our standard procedure will yield optimal paths emanating from it. For initial conditions, we can use the values of the x_i and V_i resulting from the integrals of the RPE that led to $\mathscr{T}$. (of course, V can be also computed on $\mathscr{T}$ and then employed as H there. If we were to apply the standard procedure of ascertaining the V_i on $\mathscr{T}$ in terms of the partials of H, it is not hard to see that the same V_i as above will result.[2])

On the new paths, $\bar{\phi}$, will be determined by the local sign of A. As it is zero on $\mathscr{T}$, at the outset we must rely on $\mathring{A}$. The latter can be found by a simple calculation which will actually be carried out in general in Section 7.4. It turns out that $\mathring{A}$ is independent of ϕ_1.

Now $\mathring{A}$ will be 0 on $\mathscr{T}$ only for a special class of surfaces which are detectable in advance and will occupy us extensively in Chapter 7. Therefore there exists grounds for the assumption that $\mathring{A} \neq 0$. This question will recur in better prepared context in Section 7.11.

We can conclude that when an optimal path crosses $\mathscr{T}$, where $A = 0$, $\mathring{A} \neq 0$, A changes sign and so $\bar{\phi}_1$ shifts abruptly from one of its extreme values to the other. Thus the name *transition surfaces* for such as $\mathscr{T}$.

5.2. THE DOLICHOBRACHISTOCHRONE[3]

In the classical brachistochrone problem a body in a uniform gravitational field is constrained to slide down a given curve. Its starting point is prescribed and so is its terminal position, which may be a second point or,

[2] This calculation also entails the ME_2. But as $A = 0$ on $\mathscr{T}$, it is immaterial whether we use it with $\phi_1 = a$ or b.

[3] In this chapter, where examples dominate, example and section numbers will coincide.

more generally, somewhere on a given curve. The latter alternative accords more closely with our ideas; the body is to end its fall at some point of a specified curve $\mathscr{C}$. We do not know which point and ascertaining it is part of the problem.

The issue is to find the constrained path which renders minimal the time for the body, if it starts from rest at the given starting point, to reach $\mathscr{C}$.

If from a stationary start, the body has descended a vertical distance y, no matter what its path, elementary mechanics tells us that its speed will be $\sqrt{2gy}$. Only thus does classical mechanics enter the problem. Since it is clear that the sought curve—the brachistochrone—is independent of g, the gravitational constant, we may take the speed as being $\sqrt{y}$.

In our framework, we think a point capable of being navigated about the upper half of the x, y-plane in such a way that its direction of travel at each instant is under our control, but its speed is always $\sqrt{y}$. It is clear that if we navigate so as to minimize the flight time, the situation is tantamount to the above; the optimal route will be the brachistochrone.

Thus, in the language of differential games, we take as KE

$$\dot{x} = \sqrt{y} \cos \phi$$
$$\dot{y} = \sqrt{y} \sin \phi$$

so that, as we wish, the speed of $\mathbf{x} = (x, y)$ is always $\sqrt{y}$, but travel direction of inclination ϕ is always at our—or rather P's—disposal. At each point he has a circular vectogram. Further, an integral payoff with $G = 1$ leads to the minimization of the transit time. It is trivial to observe that the usual convention of coordinates associates positive y with an upward direction; accordingly we have tacitly reversed gravity and will not be disconcerted by the "body's" falling upward.

As our terminal curve $\mathscr{C}$, we will select the positive y-axis. The playing space $\mathscr{E}$ will be the first quadrant of the plane ($x \geqslant 0$, $y \geqslant 0$). Any point of $\mathscr{E}$ can be used as a starting point, providing the "body" or point $\mathbf{x}$ is thought of as having an initial speed $= \sqrt{y}$. The positive x-axis is then the only set of starting points conforming to original framing of the problem. If the stationary start is important, we can always attain it by the proper choice of axes, but rather let us accept the modified starting rules.[4]

In the dolichobrachistochrone problem, the second player E strives to maximize the time through the addition of another velocity to those of the vectogram already discussed. He makes his selection from the vectogram shown at Figure 5.2.1a. The two extreme vectors are each of length w.

[4] Sometimes the classical problem is so stated that there is given initial speed along the curve. This is achieved by using a starting point with a suitable y.

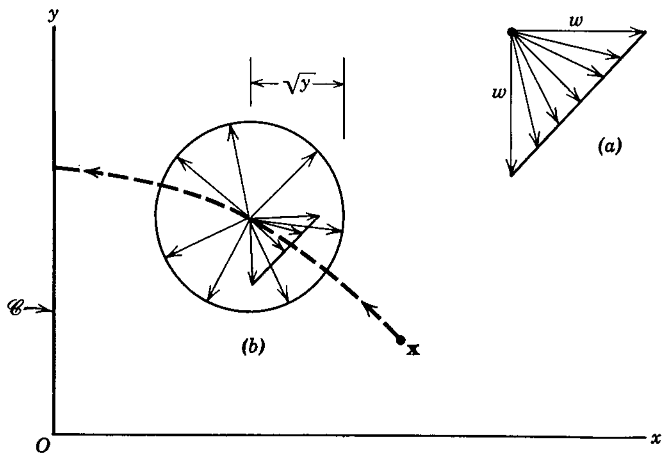


Figure 5.2.1

One gives him the chance to delay the progress of **x** toward $\mathscr{C}$ by pulling straight downward, thus impelling **x** into the region of small y and hence of low speed available to P; the other extreme vector is horizontal and directly repels **x** from $\mathscr{C}$ [(b) of the figure roughly depicts a typical instant of a partie]. The total choice open to E is a linear convex mixture of these extreme vectors and so is typical of a linear vectogram. Thus the question confronting him concerns the circumstances warranting the most profitable combination of striving directly to draw **x** away from $\mathscr{C}$ or, less directly, seeking to retard **x** by driving it deeper into the region of low speed.

Of course, there is a game of kind here too.[5] For $y < w^2$, E's speed dominates that of P and the latter cannot force termination. In fact, by a suitable alternation of his extreme vectors, it is not hard to see that E can force **x** as far away from $\mathscr{C}$ as he pleases. As far as possible, we shall ignore this aspect here.

In the following analysis, the reader may prefer to think of $w = 0$ throughout. The result will be the differential games version of the classical brachistochrone problem.

The KE are

$$\dot{x} = \sqrt{y}\cos\phi + \frac{w}{2}(\psi + 1)$$

$$\dot{y} = \sqrt{y}\sin\phi + \frac{w}{2}(\psi - 1), \qquad -1 \leqslant \psi \leqslant 1.$$

[5] It will be discussed in Example 8.6.4.

With the final terms elided, these equations have already appeared. The pair of final terms themselves are

$$(w, 0) \quad \text{when } \psi = 1$$

and

$$(0, -w) \quad \text{when } \psi = -1 \tag{5.2.1}$$

so that for the extreme allowed values of ψ we have the extreme velocities of E's vectogram. It is clear that intermediate ψ complete it in the manner described above.

As time to termination is the payoff, the latter is integral with $G = 1$. The ME_1 is accordingly

$$\min_{\phi}\max_{\psi}\left[\sqrt{y}(V_x\cos\phi + V_y\sin\phi) + \frac{w}{2}(V_x(\psi+1) + V_y(\psi-1)) + 1\right] = 0$$

The min is obtained through Lemma 2.8.1. Putting

$$\rho = \sqrt{V_x^2 + V_y^2}$$

the min is furnished by $\bar{\phi}$, where

$$\cos\bar{\phi} = -\frac{V_x}{\rho}, \qquad \sin\bar{\phi} = -\frac{V_y}{\rho}$$

and the first round parenthesis in the ME_1 becomes $-\rho$.

Since the coefficient of $\frac{1}{2}w\psi$ in the ME_1 is

$$A = V_x + V_y$$

the max is attained by

$$\bar{\psi} = \text{sgn } A.$$

Thus the ME_2 is

$$-\sqrt{y}\,\rho + \frac{w}{2}[A\bar{\psi} + (V_x - V_y)] + 1 = 0$$

or

$$-\sqrt{y}\rho + wV_x + 1 = 0 \qquad \text{when } A > 0$$

and

$$-\sqrt{y}\rho - wV_y + 1 = 0 \qquad \text{when } A < 0.$$

The RPE are, when $A > 0$ $(\bar{\psi} = 1)$,

$$\mathring{x} = \sqrt{y}\frac{V_x}{\rho} - w, \qquad \mathring{V}_x = 0$$

$$\mathring{y} = \sqrt{y}\frac{V_y}{\rho} \qquad \mathring{V}_y = -\frac{\rho}{2\sqrt{y}}.$$

When $A < 0$ $(\bar{\psi} = -1)$ the $-w$ term is removed from the equation for $\mathring{x}$ and $+w$ is added to that for $\mathring{y}$. Such is clear from a glance at the KE.

We parameterize $\mathscr{C}$ by

$$x = 0, \qquad y = s \geqslant 0$$

and on $\mathscr{C}$,

$$V(=H) = 0. \tag{5.2.2}$$

To find the useable part, we first note that the normal velocity to $\mathscr{C}$ is horizontal and so is given by $\dot{x}$ from the KE with $x = 0$, $y = s$. For penetration of $\mathscr{C}$ to be possible by P, against all opposition from E, the minimax of this velocity must < 0 and so we must have

$$\min_{\phi} \max_{\psi} \dot{x} = \min_{\phi} \max_{\psi} \left[\sqrt{s} \cos \phi + \frac{w}{2}(\psi + 1)\right]$$
$$= -\sqrt{s} + w < 0$$

or the useable part is that part of $\mathscr{C}$ for which

$$s > w^2.$$

To complete the initial conditions for integration of the RPE we must know V_x and V_y on the useable part of $\mathscr{C}$. There we have

$$V_s(= H_s) = 0 = V_x x_s + V_y y_s = V_y.$$

On $\mathscr{C}$, $V_x \geqslant 0$. For $V = 0$ on $\mathscr{C}$, and we must have $V > 0$ a little to the right of $\mathscr{C}$, as is clear if we recall that the meaning of V is the time required to reach $\mathscr{C}$. Thus $A = V_x + V_y = V_x \geqslant 0$ and so near and at $\mathscr{C}$ we may take $\bar{\psi} = +1$. The ME_2 at $\mathscr{C}$ is then (we put $V_y = 0$; as $V_x \geqslant 0, \rho = V_x$)

$$-\sqrt{s}\, V_x + w V_x + 1 = 0$$

or on $\mathscr{C}$

$$V_x = \frac{1}{\sqrt{s} - w} \tag{5.2.3}$$

which is positive on the useable part. This equation, $V_y = 0$, and (5.2.2) are the initial conditions. We employ them in the integrals of the RPE.

The upper right RPE integrates at once to (5.2.3), which is now construed as holding in at least some of $\mathscr{E}$ rather than merely on $\mathscr{C}$. The easiest way to proceed with the integration is to utilize the ME_2 (with $\bar{\psi} = 1$):

$$\sqrt{y}\, \rho = 1 + w V_x = 1 + \frac{w}{\sqrt{s} - w} = \frac{\sqrt{s}}{\sqrt{s} - w}$$

Squaring and solving for V_y leads to

$$V_y = \pm \frac{\sqrt{s/y - 1}}{\sqrt{s} - w}.$$

Which sign? A quick answer follows from observing that, for large y,

P's higher speed renders the time ($= V$) to reach $\mathscr{C}$ smaller and so $V_y < 0$. A more formal criterion we leave to the reader as

Exercise 5.2.1. Show that to satisfy the lower right RPE, the $\pm$ in V_y must be $-$.

Noting that

$$\rho = \frac{\sqrt{s}}{\sqrt{s} - w} \cdot \frac{1}{\sqrt{y}}$$

the lower left RPE becomes

$$\mathring{y} = \sqrt{y}\left(-\sqrt{\frac{s}{y} - 1}\right) \cdot \frac{1}{\sqrt{s}/\sqrt{y}} = -\sqrt{y - \frac{y^2}{s}}$$

which has as the integral with $y(0) = s$

$$y = \frac{s}{2}\left(1 + \cos \frac{\tau}{\sqrt{s}}\right) \tag{5.2.4}$$

as may be readily verified, provided that

$$\tau \leqslant \pi\sqrt{s}. \tag{5.2.5}$$

Nothing but standard methods are required to integrate the upper left RPE; the result is

$$x = \frac{\tau\sqrt{s}}{2} + \frac{s}{2} \sin \frac{\tau}{\sqrt{s}} - w\tau. \tag{5.2.6}$$

The optimal paths are given by (5.2.4) and (5.2.6). If $w = 0$, we remark that they express the classical cycloids normal to $\mathscr{C}$. The generating circle here rolls on the x-axis, has radius $= s/2$ and rotates through an angle $\tau/\sqrt{s}$ in time τ. The inequality (5.2.5), necessary for $\mathring{y}$ in the RPE to maintain the proper sign, precludes cycloidal arcs of more than a half revolution (**x** cannot reach and then leave the x-axis).

Returning to the two-player game, we have $A > 0$ and so E plays $\bar{\psi} = 1$, his horizontal extreme velocity vector. Let us see how long this state persists. We see that

$$A = V_x + V_y = \frac{(1 - \sqrt{s/y - 1})}{(\sqrt{s} - w)}$$

and A remains positive as long as $y > \frac{1}{2}s$. When $y = \frac{1}{2}s$, we should expect a transition surface (TS) where E shifts his strategy from $\bar{\psi} = 1$ to $\bar{\psi} = -1$. From (5.2.4), on the TS, $\cos \tau/\sqrt{s} = 0$ and from (5.2.5) this implies

$$\tau = \tau_0 = \frac{\sqrt{s}\,\pi}{2}. \tag{5.2.7}$$

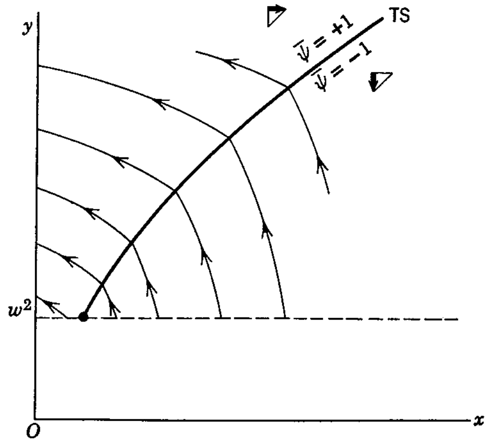


Figure 5.2.2

Then on the TS, from (5.2.6) with $\tau = \tau_0$,

$$x = \left(\frac{\pi}{4} + \frac{1}{2}\right)s - w\frac{\pi}{2}\sqrt{s}. \tag{5.2.8}$$

Note that in this domain of the first stage paths (prior to the TS)

$$V = \tau. \tag{5.2.9}$$

The equations of the TS are (5.2.8) and $y = s/2$; on it $V = \tau_0 = (\pi/2)\sqrt{s}$. It is thus a parabola with inclined axis and appears as in Figure 5.2.2. It delineates E's optimal strategy; he always plays one of his extreme velocity vectors depending on which side of the TS **x** lies, as marked on the figure.

The optimal strategy $\bar{\phi}$ of P is expressed in terms of V_x and V_y as in the derivation of the ME_2. To the left of the TS, for example,

$$\cos\bar{\phi} = -\frac{V_x}{\rho} = -\frac{\sqrt{y}}{\sqrt{s}}.$$

But a formal expression in terms of x and y is awkward. We would have to solve the path equations (5.2.4) and (5.2.6) for s and τ and use the value obtained of the former in the above formula.

The optimal paths to the left of the TS, as we can see from their equations, are generated somewhat as are cycloids. Each is described by a point on the rim of circle rolling to the right on a platform sliding to the left with speed w.

To proceed with the solution to the right of the TS requires some further integration, which, although elementary in principle, appears rather oppressive. As initial conditions we would have

$$x = \left(\frac{\pi}{4} + \frac{1}{2}\right)s + \frac{1}{2}w\pi\sqrt{s}$$

$$y = \frac{1}{2}s$$

$$V_x = -V_y = \frac{1}{\sqrt{s} - w}, \qquad s \geqslant w^2$$

and our task would be to use them in an integration of the RPE with $\bar{\psi} = -1$. The ME_2 here would be

$$-\rho\sqrt{y} - wV_y + 1 = 0.$$

There seems to be insufficient reward for the tedious formal integration required.

To establish that such a calculation actually completes the solution we would need to know that

1. the new paths cover the subset of $\mathscr{E}$ to right of the TS and for which $y > w^2$ completely and univalently.
2. $A < 0$ on this domain (except at the TS where $A = 0$).

Now 1 seems highly plausible (a proof without integration is probably not too hard). And 2 follows from 1, for along an optimal path

$$\mathring{A} = -\frac{\rho}{2\sqrt{y}} < 0.$$

5.3. THE RELATIONSHIP TO THE EULER EQUATION

The connection between our approach and the classical one, illustrated by the last example, when $w = 0$ will be discussed here in rough terms. We deal only with planar one-player games with integral payoffs and a single control variable; more elaborate cases can be inferred from this instance.

We start from the KE

$$\dot{x} = f_1(x, y, \phi)$$

$$\dot{y} = f_2(x, y, \phi)$$

with

$$\int G(x, y, \phi)\, dt \tag{5.3.1}$$

to be minimized. We can write (5.3.1) as

$$\int \frac{G(x, y, \phi)}{f_1(x, y, \phi)}\, dx. \tag{5.3.2}$$

and also

$$y'\left(=\frac{\dot y}{\dot x}\right) = \frac{f_2(x, y, \phi)}{f_1(x, y, \phi)}. \tag{5.3.3}$$

We solve (it is generally possible) (5.3.3) for ϕ and substitute the result in (5.3.2), which now assumes the familiar calculus of variations form:

$$\int F(x, y, y')\, dx. \tag{5.3.4}$$

The case where F is independent of y' plays a special part in our theory and will reappear during our later discussion of universal surfaces.

On the other hand, if we start from (5.3.4), there are many ways of reaching the KE and (5.3.1) inasmuch as (5.3.4) takes no account of the t-parameterization. The simplest appears to be

$$\dot x = \cos \phi$$
$$\dot y = \sin \phi$$
$$G = F(x, y, \tan \phi) \cos \phi.$$

5.4. THE WAR OF ATTRITION AND ATTACK[6]

The two nations, P and E, engaged in a protracted war, have respective supplies of a vital weapon, x_1 and x_2, at time t. Each has at all times the choice of how to allocate his stock between "*attrition*," that is, depleting his enemy's rate of weapon supply, and "*attack*," that is, entering them in the major conflict. It is the accumulation of the latter entries that count; each player seeks more than his opponent, and the excess will be the payoff.

Thus the basic decisions here are between the long-range policy of attrition and the short-range one of direct attack on the essential targets. We shall formulate what appears to be the simplest version possible of this broad problem. It will be discussed further in Chapter 11, where we shall frame an alternative model in Section 11.9.

The realistic execution of the strategies would comprise a series of discrete decisions. But we shall smooth matters into a continuous process. Such is certainly no farther from truth than our assumptions and therefore can be expected to be as reliable as a stepped model. It also is more facile to handle and yields general results more readily.

[6] Proposed by Arnold Mengel.

Let P, at time t, split his force x_1 into the attacking component $(1 - \phi)x_1$ and that of attrition ϕx_1. Here ϕ, the fraction devoted to the latter purpose, satisfies $0 \leqslant \phi \leqslant 1$. If E is unimpeded he has the capacity to manufacture weapons at the rate m_2. He also loses them at a rate depending on the number ϕx_1 his enemy is devoting to that purpose. For lack of better information, let us take this rate as proportional to the number. Then we may write

$$\dot x_2 = m_2 - c_2(\phi x_1)$$

where the coefficient c_2 may be regarded as a measure of effectiveness of P's weapons against E's defenses.

By reversing the roles of the players we obtain a second similar equation; they will be two of the KE.

Let us suppose we plan on the war lasting some definite time T.[7] Each day (let us say) the combatants put into the field the numbers of weapons $(1 - \phi)x_1$ and $(1 - \psi)x_2$. The sums of these quantities for each day will reflect the respective sides' total battle strength, and difference of such sums, the margin of superiority. Such will be the payoff except that, in accordance with our policy of smoothing, the sum becomes an integral:

$$\int_0^T [(1 - \psi)x_2 - (1 - \phi)x_1]\, dt.$$

To fit the framework of differential games we adopt T (or x_3, if preferred) as a state variable. Then the totality of the KE is

$$\dot x_1 = m_1 - c_1 \psi x_2$$
$$\dot x_2 = m_2 - c_2 \phi x_1$$
$$\dot T = \dot x_3 = -1$$

with $0 \leqslant \phi, \psi \leqslant 1$ and $G = (1 - \psi)x_2 - (1 - \phi)x_1$.

The space $\mathscr{E}$ will be the octant

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad T \geqslant 0$$

and $\mathscr{C}$ will be its partial boundary where $T = 0$, which may be parameterized as

$$x_1 = s_1 \geqslant 0, \quad x_2 = s_2 \geqslant 0, \quad T = 0$$

these equations being part of the initial conditions.

We shall also suppose

$$c_1 > c_2 \tag{5.4.1}$$

reversing the notation if the contrary were true.[8]

[7] Alternatives to such assumptions are suggested in Chapter 11.

[8] In this type of problem we prefer to eschew such symmetries as $c_1 = c_2$ as being too coincidental.

For the time being, we shall ignore the possibility of x_1 or x_2 becoming negative. In this way we can get to the heart of the problem with less encumbrance.

Denoting the partials of V by V_1, V_2, V_T, we can write the ME_1:

$$\min_{\phi}\max_{\psi}\,[(m_1 - c_1\psi x_2)V_1 + (m_2 - c_2\phi x_1)V_2 - V_T + (1-\psi)x_2 - (1-\phi)x_1] = 0$$

and the ME_2 is then

$$S_1x_1\bar\phi + S_2x_2\bar\psi + m_1V_1 + m_2V_2 - V_T + x_2 - x_1 = 0$$

where $S_1 = 1 - c_2V_2,\quad S_2 = -1 - c_1V_1$

and $$\bar\phi = \begin{cases}0 \text{ if } S_1 > 0\\ 1 \text{ if } S_1 < 0\end{cases},\qquad \bar\psi = \begin{cases}0 \text{ if } S_2 < 0\\ 1 \text{ if } S_2 > 0.\end{cases}$$

Observe the tacit assumption that $x_1, x_2 \geqslant 0$.

The RPE follow:

$$\mathring{x}_1 = -m_1 + c_1\bar\psi x_2,\qquad \mathring{V}_1 = S_1\bar\phi - 1$$
$$\mathring{x}_2 = -m_2 + c_2\bar\phi x_1,\qquad \mathring{V}_2 = S_2\bar\psi + 1$$
$$\mathring{T} = 1$$

the expression for $\mathring{V}_T$ turning out to be superfluous in this particular game.

Let us complete the initial conditions. As on $\mathscr{C}$ we have $V = 0$,

$$\frac{\partial V}{\partial s_1} = 0 = V_1\frac{\partial x_1}{\partial s_1} + V_2\frac{\partial x_2}{\partial s_1} + V_T\frac{\partial T}{\partial s_1} = V_1$$

and, similarly, $V_2 = 0$ on $\mathscr{C}$. There, then,

$$S_1 = 1 \text{ and so } \bar\phi = 0$$
$$S_2 = -1 \text{ and so } \bar\psi = 0.$$

Thus the war concludes with both sides fully attacking.

We now integrate the RPE, using the above initial conditions with $\bar\phi = \bar\psi = 0$.

$$\begin{aligned} x_1 &= s_1 - m_1\tau & V_1 &= -\tau\\ x_2 &= s_2 - m_2\tau & V_2 &= \tau\\ T &= \tau. \end{aligned}\tag{5.4.2}$$

On the optimal paths

$$S_1 = 1 - c_2\tau,\qquad S_2 = -1 + c_1\tau$$

and these first cease to be positive when $\tau = 1/c_2$ and $1/c_1$. Because of

(5.4.1) the latter occurs first (retrogressively). Thus we should expect a TS (transition surface) when

$$(\tau =)\ T = \frac{1}{c_1}\tag{5.4.3}$$

where $\bar\psi$ should change from 0 to 1. Such is the case; continuation of the analysis furnishes a proof. Then at a time $1/c_1$ short of the end of the war, E switches from full attrition to full attack.

Let the surface (5.4.3) be called $\mathscr{T}_1$. The Value between $\mathscr{C}$ and $\mathscr{T}_1$ is given by

$$V = \int_0^T G\,dt = \int_0^T (x_2 - x_1)\,dt = \int_0^T [(s_2 - m_2\tau) - (s_1 - m_1\tau)]\,d\tau$$
$$= (s_2 - s_1)T - \tfrac{1}{2}(m_2 - m_1)T^2.\tag{5.4.4}$$

To express V in terms of the state variables, we put $\tau = T$ in (5.4.2) on the left and eliminate s_1, s_2. We find that

$$V = (x_2 - x_1)T + \tfrac{1}{2}(m_2 - m_1)T^2.\tag{5.4.5}$$

To continue our analysis, we treat $\mathscr{T}_1$ as a seat of initial conditions in the same manner as we did $\mathscr{C}$. We may either place $\tau = 1/c_1$ in (5.4.2) and again use s_1 and s_2, but now as parameters of $\mathscr{T}_1$, or we may start afresh with new parameters. Choosing the latter alternative, $\mathscr{T}_1$ is

$$x_1 = s_1,\quad x_2 = s_2,\quad T = \frac{1}{c_1}$$

where these s_i are not the same as the old. From (5.4.2) on the right we have at once that on $\mathscr{T}_1$,

$$V_1 = -\frac{1}{c_1},\qquad V_2 = \frac{1}{c_1}.$$

Of course these two conditions could also be derived in our standard way by using (5.4.5) on $\mathscr{T}_1$ as H.

When integrating the RPE with the above starting data, we shall use $\phi = 0$, $\psi = 1$. This usage will be confirmed if it leads to S_i of the proper signs. The reader may find it beneficial to perform the calculation which results in

$$\begin{aligned} x_1 &= s_1 + (c_1s_2 - m_1)\tau - \tfrac{1}{2}c_1m_2\tau^2\\ x_2 &= s_2 - m_2\tau\\ T &= \frac{1}{c_1} + \tau \end{aligned}\tag{5.4.6}$$

$$V_1 = -\frac{1}{c_1} - \tau,\quad V_2 = \frac{1}{c_1} + \tau + \frac{1}{2}c_1\tau^2.\tag{5.4.7}$$

Here τ, as the s_i, was chosen afresh as $\mathscr{T}_1$ and is not the same as in the earlier integration. We find now that

$$S_1 = 1 - \frac{c_2}{c_1} - c_2\tau - \frac{1}{2}c_1c_2\tau^2$$
$$S_2 = c_1\tau.$$

Clearly $S_2 > 0$ when $\tau > 0$, and we are confirmed in our expectation that $\bar{\psi} = 1$.

When $\tau = 0$, because of (5.4.1), $S_1 > 0$ and so $\phi = 0$ for small τ. But this condition will cease should S_1 stop being positive. The equation, $S_1 = 0$, has just one positive root, which is

$$\tau_2 = \frac{1}{c_1}[-1 + \sqrt{(2c_1/c_2) - 1}].$$

If we accept this value as marking a second TS and also (very plausibly) assume there will be no further shifts in the control variables, we have fully found the optimal strategies in those cases (of most practical interest) where x_1 and x_2 do not become zero.

At time $1/c_1$ short of the duration E shifts from attrition to attack and P does so earlier, at a time preceding termination by

$$\frac{1}{c_1}\sqrt{2(c_1/c_2) - 1}. \tag{5.4.8}$$

The last value arises, of course, because of the new TS, which we will call $\mathscr{T}_2$ when $T = (5.4.8)$.

Let us now find V between the $\mathscr{T}_1$ and $\mathscr{T}_2$. We can do so in two ways. First we can write

$$V = V^{(1)} - \int_0^{T-1/c_1} x_1\,dt$$

where $V^{(1)}$ is V on $\mathscr{T}_1$ obtained from (5.4.5) with arguments from (5.4.6). The integrand x_1 (= G) is also taken from (5.4.6), and finally we use these equations to eliminate s_1 and s_2.

The second way is to take V_1 and V_2 from (5.4.7), replacing τ by $T - 1/c_1$, and then getting V_T from the ME_2. Integrating these partials gives V to within an additive constant, which can be ascertained by its known value on $\mathscr{T}_1$.

The result is

$$V = \frac{x_2}{2c_1} - \frac{m_2}{6c_1^{\,2}} + \left(\frac{m_2}{2c_2} - x_1\right)T + \tfrac{1}{2}(c_1x_2 - m_1)T^2 - \frac{c_1m_2T^3}{6}. \tag{5.4.9}$$

Observe that beyond $\mathscr{T}_2$, $G = 0$ and there are no further changes in V. It is then given by (5.4.9) with T fixed at (5.4.8).

We can now accept on heuristic grounds that $\mathscr{T}_2$ is a TS and there will be no further shifts in strategy, or use the criterion to come in Section 7.11, or continue with the solution beyond $\mathscr{T}_2$. We can then see directly whether there are further sign changes in S_1 and S_2. We also obtain the optimal paths in this domain.

We leave the decision and, in the latter case the effort, to the reader.

There remains the embodiment of the constraints $x_i \geqslant 0$ into the solution. The best method seems to consist of first treating the boundary parts of $\mathscr{E}$, where $x_i = 0$ as 2-dimensional subgames; when the Values of these are known, they can be used as H and new optimal paths into $\mathscr{E}$ can be constructed which may be merged with the old.

More specifically, let $\mathscr{C}_1$ be the subset of the boundary of $\mathscr{E}$ described and parameterized by

$$x_1 = 0, \quad x_2 = s_2 \geqslant 0, \quad T = s_3 \geqslant 0$$

and $\mathscr{C}_2$ is defined similarly with the subscripts 1 and 2 interchanged.

Now when the state of a partie (that is, $\mathbf{x}$) is on $\mathscr{C}_1$, P has no weapons ($x_1 = 0$), and so it would be futile for E to induce more attrition (take ψ larger) than is needed to keep x_1 at 0. It is of course futile in practice because E would be wasting weapons on attrition that could be used for attack and so score in the payoff; but in theory, when we view the game as a purely mathematical problem, we might regard such a limitation on ψ as a *given* constraint necessary to keep $\mathbf{x}$ within the bounds of $\mathscr{E}$.

Let us consider the part of $\mathscr{C}_1$ or $\mathscr{C}_2$ lying between $\mathscr{C}$ and $\mathscr{T}_1$, a region where we know that $\bar{\phi} = \bar{\psi} = 0$. From the KE, $\dot{x}_1 > 0$, $\dot{x}_2 > 0$ (intuitively: the weapon numbers increase with no attrition); therefore at these points the optimal paths leave $\mathscr{C}_1$ and $\mathscr{C}_2$ (progressively) and present no problem.

Figure 5.4.1 is a typical cross section of $\mathscr{E}$ for some constant fairly large x_2, with the optimal paths indicated. The paths found earlier between $\mathscr{C}$ and $\mathscr{T}_1$ appear as those above AB; those below have just now been discussed. If they are traced back (increasing τ), they each reach $\mathscr{C}_1$ at some point of OA and induce no modification to our analysis.

But on $\mathscr{C}_1$ beyond $\mathscr{T}_1$, where $\bar{\psi} = 1$, it can be that x_1 remains 0 during an interim. There can be paths like $CDEAB$ on the figure. Recalling the KE

$$\dot{x}_1 = m_1 - c_1\psi x_2$$

we see that x_1 can remain 0 (such as on EA, as just illustrated) only if

$$x_2 \geqslant \frac{m_1}{c_1}. \tag{5.4.10}$$

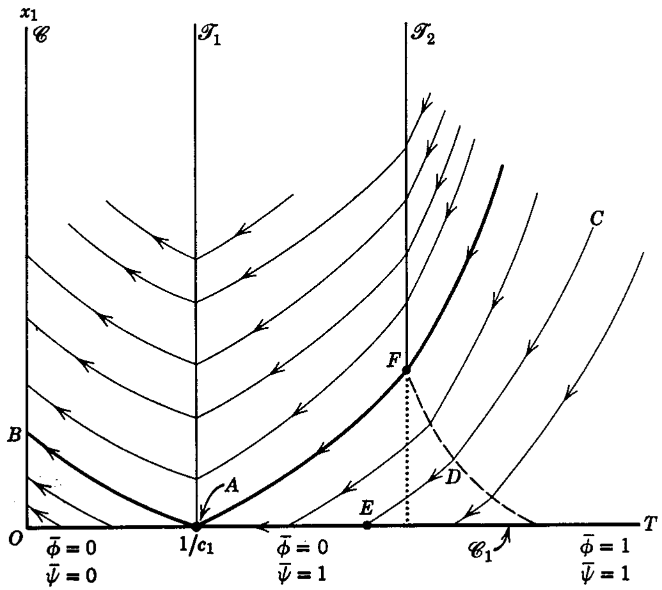


Figure 5.4.1

When this is so, and $x_1 = 0$, E instead of playing $\psi = \bar{\psi} = 1$, should employ

$$\psi = \frac{m_1}{c_1 x_2} \tag{5.4.11}$$

until $\mathbf{x}$ reaches $\mathscr{T}_1 \cap \mathscr{C}_1$ (A in the figure). For this phase we set up the subgame as follows:

The state variables are x_2 and T (as $x_1 = 0$); the temporary $\mathscr{E}$ is subject to (5.4.10) and $T > 1/c_1$. (It appears as the hatched region on the "floor" of Figure 5.4.2, which depicts the $\mathscr{E}$ of the original total game.) The temporary $\mathscr{C}$ is given by

$$x_2 = \sigma \text{ (a parameter)} \geqslant \frac{m_1}{c_1}$$

$$T = \frac{1}{c_1}$$

and H is the Value (5.4.5) at these points:

$$H = \frac{\sigma}{c_1} + \frac{(m_2 - m_1)}{2c_1^{\,2}}.$$

The KE are

$$\dot{x}_2 = m_2$$
$$\dot{T} = -1$$

and for G, in virtue of (5.4.11) and $x_1 = 0$, we have

$$G = \left(1 - \frac{m_1}{c_1 x_2}\right) x_2 = x_2 - \frac{m_1}{c_1}.$$

We leave the details of this simple subgame, with no control variables, to the reader. All we will utilize of its solution is, as yielded by certain integrals of the RPE,

$$V_2 = T$$
$$V_T = -\frac{m_1}{c_1} + x_2 + m_2 T. \tag{5.4.12}$$

Now we return to the original game. We are to construct regressive paths with partial initial conditions on part of $\mathscr{C}_1$:

$$x_1 = 0, \quad x_2 = s_2 \geqslant \frac{m_1}{c_1}, \quad T = s_3 \geqslant \frac{1}{c_1}.$$

Putting these values into (5.4.12), we obtain further initial conditions. The remaining V_1 is obtained from the ME_2 with $\phi = 0$, $\psi = 1$, the result being

$$V_1 = -\frac{1}{c_1}.$$

The integration of the RPE with the above initial conditions is done as earlier in the section. Similarly, as earlier, we investigate sign changes of the S_1 along these paths to locate TS.

We shall omit the calculations. It turns out that S_2 remains positive, but S_1 changes sign, but not on $\mathscr{T}_2$. Rather, part of this plane has to be replaced by a curved surface.

In Figure 5.4.1 the new paths appear as AF and others below it, such as ED. Below F, $\mathscr{T}_2$ is to be replaced by the section shown dashed, where S_1 changes sign. The full picture is seen in Figure 5.4.2. The "seam," where the new curved surface replaces the old $\mathscr{T}_2$, is found by considering paths emanating from AB.

What means this modified $\mathscr{T}_2$?

Broadly, the solution we have found tells us that, for T reasonably large, both players begin the partie with all their forces devoted to the long-range objective of attrition. Then, at a certain critical time for each

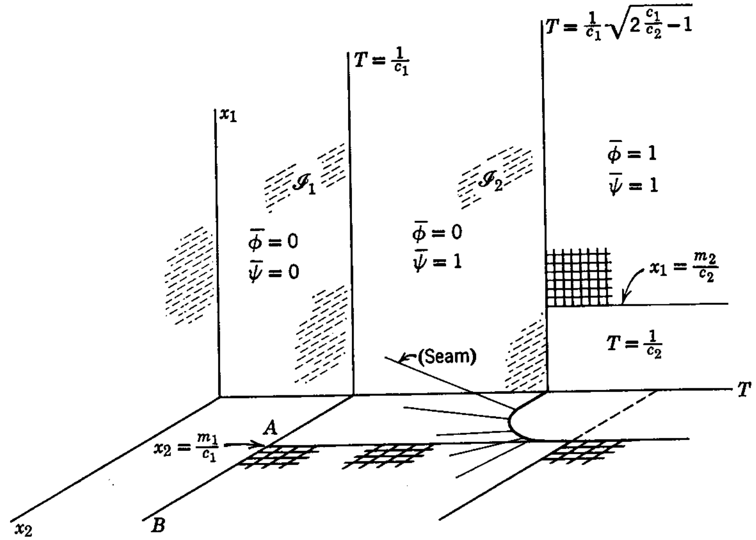


Figure 5.4.2

before termination, they switch to all-out attack. These times essentially depended on the c_i only; the forces x_1, x_2 and manufacturing rates m_1, m_2 do not matter. The curved part of $\mathscr{T}_2$ is an exception. It means that P does best by switching from attrition to attack earlier, at a time depending on x_1 and x_2. We seem to have a mathematical confirmation of the reasoning by P: "If the later part of my attack effort is going to be nullified through my forces being annihilated, I had better compensate by starting it sooner."

There is a similar phenomenon on $\mathscr{C}_2$, where x_2 can become temporarily zero. It occurs in the hatched area on the vertical "wall" of Figure 5.4.2. Paths emanate from here into $\mathscr{E}$ but do not affect the TS.

Finally, note that the formulas given for V do not hold in the domains covered by the paths emanating from either $\mathscr{C}_i$.

Exercise 5.4.1. Find the equations of the curved part of $\mathscr{T}_2$. In particular, show that, as suggested by the figure, sections of constant T are straight lines.

Exercise 5.4.2. Analyze the situation regarding the paths emanating from $\mathscr{C}_2$.

Exercise 5.4.3. For both new classes of paths, find V and compare it to the old.

5.5. THE ISOTROPIC ROCKET PURSUIT GAME

The pursuer P is driven by a thrust of fixed magnitude F,[9] but whose direction he can control; in this way he navigates. The evader E has simple motion with the fixed speed w. The action takes place in the plane and the payoff is the time to capture.

Of course, without a gain of difficulty in principle, we could assign more complex and realistic kinematics to E, but in practice we would be encumbered with more state variables, more elaborate KE and a much more tedious analysis. If the kinematics in the first paragraph were retained, but the roles of P and E reversed, we would have a problem of about the same difficulty as, and in many ways analogous to, the present one. If we permitted intermediate navigational magnitudes, that is, allowed E speeds $\leqslant w$ and P thrusts $\leqslant F$, nothing would be gained; for in optimal play (the reader can verify this) both players would employ their extreme values at all times.

This problem is similar in broad outline to the homicidal chauffeur game. But the solution to the latter, although the local motion is usually very simple, entails numerous singular surfaces and will have to await more advanced theory. Here the differential aspects of the motion are more intricate but the solution is much more analytic; a single integration of the RPE suffices for an entire optimal path. Formally one might say that here we have no linear vectograms; all max and min are interior; there are no abrupt changes to and from extreme values which can be subtly taxing. Nevertheless we will find here a counterpart of the swerve maneuver (Section 1.5).

We will also burden P with a friction drag taken as negatively proportional to his velocity. There were two reasons. One was that without the drag there is no bound on P's speed and, although not too much realism is claimed for this partially idealized problem, completely uncurbed speeds seem too shocking a transgression. If the friction force is $-k$ times the speed, there is, as is well known, a natural limit to the latter equal to F/k. It is the speed P would come to asymptotically if his thrust propelled him along a straight line. The second reason is the interesting question of what circumstances enable P to capture at all if $w > F/k$. Although such games of kind come later (Chapters 8 and 9), we will be able to infer the answer here.

If the reader prefers to elide this complication of friction, he can do so by imagining the k in our analysis below always replaced by 0. We shall

[9] More accurately F is the thrust per unit mass of P or specific thrust.

indicate the places where such engenders a formal deviation of the analysis by square brackets.

There is to be a capture radius $l > 0$, that is, capture is defined as occurring when $|PE| \leqslant l$.

We are going to begin our analysis in the realistic space which has six natural dimensions. Later we will shift to a reduced $\mathscr{E}$ with $n = 3$, which is minimal. However, the former space makes the analysis easier (if lengthier to write) and its interpretation more transparent.

To describe a state of P we need to know both his location and (vectorial) velocity, requiring four coordinates. Two, those of location, suffice for E. In the standard Cartesian plane let x, y be the coordinates of P and u, v the components of his velocity. The coordinates of E will be x_E and y_E. The control variable of the former player will be ϕ, the angle his thrust vector makes with the y-axis. For E, ψ will be the inclination of his travel direction to this axis.

The KE are then

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= v\\ \dot{u} &= F\sin\phi - ku\\ \dot{v} &= F\cos\phi - kv\\ \dot{x}_E &= w\sin\psi\\ \dot{y}_E &= w\cos\psi.\end{aligned}$$

The first two merely say that the rate of change of P's affix is his velocity. The next pair equate his acceleration to the propelling specific thrust less the drag. The last two relate to E's simple motion.

As time of capture is the payoff, $G = 1$.

We shall write the partial of V with respect to x_E as V_{xE} and similarly for y_E. We have then for the ME_1,

$$uV_x + vV_y + \min_\phi F(V_u\sin\phi + V_v\cos\phi) - k(uV_u + vV_v) + \max_\psi w(V_{xE}\sin\psi + V_{yE}\cos\psi) + 1 = 0.$$

Putting

$$\rho = \sqrt{V_u^2 + V_v^2}$$

and

$$\rho_E = \sqrt{V_{xE}^2 + V_{yE}^2}$$

we have, from Lemma 2.8.1,

$$\sin\bar{\phi} = -\frac{V_u}{\rho}, \qquad \cos\bar{\phi} = -\frac{V_v}{\rho}$$

$$\sin\bar{\psi} = \frac{V_{xE}}{\rho_E}, \qquad \cos\bar{\psi} = \frac{V_{yE}}{\rho_E}.$$

Thus we obtain for the ME_2

$$uV_x + vV_y - F\rho - k(uV_u + vV_v) + w\rho_E + 1 = 0.$$

Our standard procedure leads to the RPE:

$$\begin{aligned}\mathring{x} &= -u, & \mathring{V}_x &= 0\\ \mathring{y} &= -v, & \mathring{V}_y &= 0\\ \mathring{u} &= F\frac{V_u}{\rho} + ku, & \mathring{V}_u &= V_x - kV_u\\ \mathring{v} &= F\frac{V_v}{\rho} + kv, & \mathring{V}_v &= V_y - kV_v\\ \mathring{x}_E &= -w\frac{V_{xE}}{\rho_E}, & \mathring{V}_{xE} &= 0\\ \mathring{y}_E &= -w\frac{V_{yE}}{\rho_E}, & \mathring{V}_{yE} &= 0.\end{aligned}$$

Now let us turn our attention to $\mathscr{C}$. It is the surface in $\mathscr{E}$ characterized by $|PE| = l$. We depict it by five parameters

$$\begin{aligned}x &= s_1\\ y &= s_2\\ u &= s_3\\ v &= s_4\\ x_E &= s_1 + l\sin s_5\\ y_E &= s_2 + l\cos s_5.\end{aligned}$$

Here the coordinates germane to P are labeled freely, but E is picked a distance l away from (s_1, s_2) so that the vector EP has an inclination with the vertical equal to s_5.

To find the useable part, put $r = |PE|$ so that

$$r^2 = (x_E - x)^2 + (y_E - y)^2.$$

Then for the points of $\mathscr{C}$

$$l\dot{r} = (l\sin s_5)(w\sin\psi - u) + (l\cos s_5)(w\cos\psi - v).$$

The useable part, being specified by

$$\max_\psi \dot{r} < 0$$

is here those points of $\mathscr{C}$ for which

$$w - s_3 \sin s_5 - s_4 \cos s_5 < 0. \tag{5.5.1}$$

Exercise 5.5.1. With a vector diagram obtain this result geometrically. If the capture region is thought of as a disk centered at P, interpret the useable realistically (P can only capture with the forward portion, depending on his speed, etc.)

Let us complete the initial conditions by finding the V_i on $\mathscr{C}$. As here, clearly $V(= H) = 0$, we have

$$\frac{\partial V}{\partial s_1} = 0 = V_x + V_{xE}$$
$$V_{s_2} = 0 = V_y + V_{yE}$$
$$V_{s_3} = 0 = V_u$$
$$V_{s_4} = 0 = V_v$$
$$V_{s_5} = 0 = (V_{xE} \cos s_5 - V_{yE} \sin s_5).$$

From the last equation and then the first two, for some λ,

$$\begin{aligned} -V_x &= V_{xE} = \lambda \sin s_5 \\ -V_y &= V_{yE} = \lambda \cos s_5. \end{aligned} \tag{5.5.2}$$

If we now substitute these V_i into the ME_2 we will have an equation for λ. There will be two solutions corresponding to contact with $\mathscr{C}$ (if considered as a circle about P) from inside and outside. Here we are interested only in the latter.

Let us take $s_5 = 0$. For **x** on $\mathscr{C}$, then we will have E a distance l directly above P. Move E up slightly (increase y_E); then V will become positive. Therefore $V_{yE} > 0$ when $s_5 = 0$. The last equation above shows that

$$\lambda > 0.$$

Substituting now into the ME_2 yields

$$-\lambda(s_3 \sin s_5 + s_4 \cos s_5) + w\lambda + 1 = 0$$

(Observe that on $\mathscr{C}$, $\rho = 0$, but $\rho_E = |\lambda|$; it was because of the latter that we needed to know sgn λ.)

Thus

$$\lambda = \frac{1}{s_3 \sin s_5 + s_4 \cos s_5 - w}$$

which is positive on the useable part by (5.5.1).

We are now ready to integrate. Treating the V_i first we at once see that (5.5.2) holds along the optimal paths (not merely on $\mathscr{C}$) because from the RPE the relevant $\mathring{V}_i$ are all 0. The third and fourth right RPE integrate to

$$V_u = -\lambda(\sin s_5)\frac{1 - e^{-k\tau}}{k}$$
$$V_v = -\lambda(\cos s_5)\frac{1 - e^{-k\tau}}{k}.$$

[If $k = 0$, these become $V_u = -\lambda\tau \sin s_5$, $V_v = -\lambda\tau \cos s_5$.]

At this point we can profitably return to the optimal strategies. Noting that

$$\rho = \lambda \frac{1 - e^{-k\tau}}{k} \, [= \lambda\tau \text{ if } k = 0]$$
$$\rho_E = \lambda$$

we find that

$$\sin \bar{\phi} = \sin s_5, \qquad \cos \bar{\phi} = \cos s_5$$
$$\sin \bar{\psi} = \sin s_5, \qquad \cos \bar{\psi} = \cos s_5$$

or

$$\bar{\phi} = \bar{\psi} = s_5.$$

Here is our first significant result. Under optimal play both control variables will be constant and equal, that is, E flies a straight line and P maintains his thrust vector in a constant direction.[10] Further, this direction is the same as that of E's path. The final equalities state that at capture P will be directly behind E (in his flight direction sense).

Problem 5.5.1. How much of this result depends on the KE and how much on $\mathscr{C}$? How would it be changed with a different type of termination (say a less regular capture region than a circle)? See the missile trajectory problem in the Appendix for such a discussion of a related problem.

The quantitative problem of the optimal strategies is now reduced to finding the above mutual direction. It will of course be a function defined over $\mathscr{E}$.

We now integrate the left RPE. The results are, as the reader may easily check,

$$\begin{aligned} x &= s_1 - s_3\left(\frac{e^{k\tau} - 1}{k}\right) + F(\sin s_5)\frac{e^{k\tau} - 1 - k\tau}{k^2} \\ u &= s_3 e^{k\tau} - F(\sin s_5)\frac{e^{k\tau} - 1}{k} \\ x_E &= s_1 + (l - w\tau)\sin s_5 \end{aligned} \tag{5.5.3}$$

[10] With no friction ($k = 0$), P's path then lies on a parabola.

and y, v, y_E have similar expressions except that $\sin s_5$ is replaced by $\cos s_5$, and s_1, s_3 by s_2, s_4.
[If $k = 0$, the first two expressions become

$$x = s_1 - s_3\tau + \tfrac{1}{2}F\tau^2 \sin s_5$$
$$u = s_3 - F\tau \sin s_5].$$

The next step in a full formal solution is to solve these six equations of the state variables for the six unknowns, $s_1, \ldots, s_5, \tau$. In particular, the resulting $\tau\,(x, y, \ldots, y_E)$ or $\tau(\mathbf{x})$ will be V.

If we define

$$Q(\tau) = l - w\tau + F\,\frac{e^{-k\tau} - 1 + k\tau}{k^2} \tag{5.5.4}$$

a short calculation shows that

$$x_E - x - \left(\frac{1 - e^{-k\tau}}{k}\right)u = Q(\tau)\sin s_5 \tag{5.5.5}$$

and similarly

$$y_E - y - \left(\frac{1 - e^{-k\tau}}{k}\right)v = Q(\tau)\cos s_5 \tag{5.5.6}$$

[For $k = 0$, the above becomes

$$Q(\tau) = l - w\tau + \tfrac{1}{2}F\tau^2$$
$$x_E - x - u\tau = Q \sin s_5, \text{ etc.}]$$

We can now eliminate s_5 by squaring and adding (5.5.5) and (5.5.6). To do so leads us naturally to the type of coordinates that we might have used in a reduced $\mathscr{E}$.

Let $\mathbf{r}$ be the vector $(x_E - x, y_E - y)$, the displacement of E relative to P, and $\mathbf{u}$, the velocity vector (u, v). Then when we square and add, we obtain

$$\mathbf{r}^2 - 2(\mathbf{r}\cdot\mathbf{u})\text{[11]}\left(\frac{1 - e^{-k\tau}}{k}\right) + \mathbf{u}^2\left(\frac{1 - e^{-k\tau}}{k}\right)^2 = Q(\tau)^2 \tag{5.5.7}$$

an equation that is to be solved for $\tau = V$.
[For $k = 0$, (5.5.7) is

$$\mathbf{r}^2 - 2(\mathbf{r}\cdot\mathbf{u})\tau + \mathbf{u}^2\tau^2 = Q^2.]$$

Observe that Q is a fixed function for a definite game; it does not depend on the state variables but entails only parameters. A plot of Q for

$$F = 3, \quad w = 2, \quad l = 1, \quad k = 1 \tag{5.5.8}$$

is given in Figure 5.5.1.

[11] A scalar (inner, dot) product.

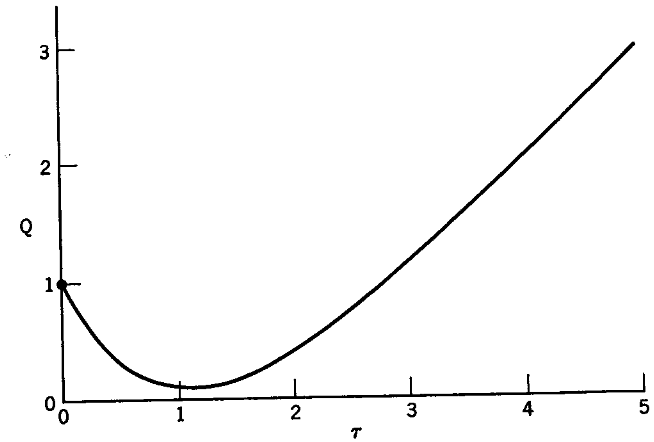


Figure 5.5.1

The left side of (5.5.7) entails all the state variables that would appear in proper reduced $\mathscr{E}$, as we shall soon demonstrate explicitly.

Let us make a tentative assumption whose significance will be discussed later:

$$Q(\tau) > 0 \qquad \text{for all } \tau \geqslant 0. \tag{5.5.9}$$

From (5.5.4) we see that

$$Q(\tau) \sim \left(\frac{F}{k} - w\right)\tau, \qquad \tau \to \infty \tag{5.5.10}$$

so that our assumption (5.5.9) requires that $F/k > w$[12] or P's limiting speed exceeds that of E.

Note that the left side of (5.5.7) is bounded for $\tau \geqslant 0$. Further, it is always positive. For, regarded as a quadratic, its discriminant is negative by the Schwarz inequality.

We see that for any interior point of $\mathscr{E}$, (5.5.7) will be satisfied for some positive τ. For when $\tau = 0$,

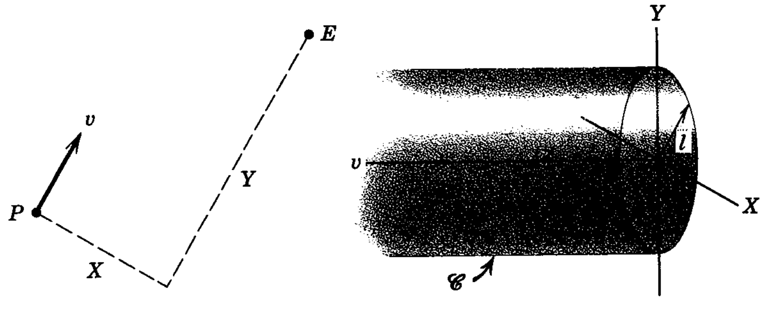
$$\text{left side} = \mathbf{r}^2 > l^2 = Q(0)^2 = \text{right side}$$

and, for large τ, this inequality, as the preceding lines show, will be reversed. Let $\tau_0 = \tau_0(\mathbf{x}) = \tau_0(x, y, \ldots, y_E)$ be the smallest such solution.

[For $k = 0$, (5.5.7) is an algebraic equation of degree four. As above, inequalities for large and small τ, show the existence of a positive root.]

This step of solving the transcendental [algebraic] equation (5.5.7) essentially completes the computational solution of the problem. For

[12] Let us spare ourselves the coincidental case of $F/k = w$.



(a) (b)

Figure 5.5.2

$\tau_0(\mathbf{x})$ itself will be $V(\mathbf{x})$. Putting $\tau_0(\mathbf{x})$ for τ in (5.5.5) and (5.5.6) enables us to find a unique function, $s_5(\mathbf{x})$ defined throughout $\mathscr{E}$. The optimal strategies are

$$\bar{\phi} = \bar{\psi} = s_5(\mathbf{x}) \tag{5.5.11}$$

We shall leave much of the formal verification to the reader. If a partie starts from $\mathbf{x}$ and the players use the strategies (5.5.11), holding them constant during play, he can verify that the function τ_0, calculated at each succeeding position, decreases at a unit rate. After a time τ_0, then, τ_0 will be zero (this is why we must use the smallest root) and r will be l and be so for the first time.

But it is much more enlightening to translate into the tongue of a reduced space. The new $\mathscr{E}$ will have three dimensions, and we can envisage the whole action clearly.

Let P and E be as shown in Figure 5.5.2a with the arrow denoting P's velocity, which has magnitude v.[13] The relative coordinates, X and Y, shown for E are measured parallel and normal to this vector. We take X, Y, v as coordinates in our reduced space. In it $\mathscr{C}$ appears as a cylinder of radius l centered on the v-axis (Equation: $X^2 + Y^2 = l^2$), and $\mathscr{E}$ is defined as the half-space (X, Y, v with $v \geqslant 0$) deprived of the interior of this cylinder as shown at (b) of the figure.

To translate (5.5.7) into these terms we observe that

$$\mathbf{r}^2 = X^2 + Y^2$$
$$\mathbf{r} \cdot \mathbf{u} = vY$$
$$\mathbf{u}^2 = v^2$$

[13] Not to be confused with the old v, which we shall not use again.

so that (5.5.7) now appears as

$$X^2 + \left[Y - v\left(\frac{1 - e^{-k\tau}}{k}\right)\right]^2 = Q^2(\tau). \tag{5.5.12}$$

[or $X^2 + (Y - v\tau)^2 = Q^2$ when $k = 0$.]

We shall now examine the surfaces (5.5.12) in $\mathscr{E}$ of constant τ. If, for the moment, we also fix v, (5.5.12) is the equation of a circle with center at $X = 0$, $Y = v(1 - e^{-k\tau})/k$ and radius Q. Thus the surface we seek is a cylinder of this "radius" whose vertical sections are circles and whose axis is the line

$$X = 0, \quad Y = \lambda\left(\frac{1 - e^{-k\tau}}{k}\right), \quad v = \lambda, \qquad 0 \leqslant \lambda \leqslant \infty.$$

When $\tau = 0$, note that this cylinder becomes $\mathscr{C}$.

Figure 5.5.3 is a rough sketch of how the family of cylinders begins, that is, with τ small. Figure 5.5.4 is a carefully drawn cross section at $v = 2.5$ for the particular data (5.5.8).

Observe how the circles of constant $\tau (= V)$ for low values have a prominently visible envelope. The two curves of this envelope meet $\mathscr{C}$ at the boundary of the useable part. Our rejection of all but the smallest solution of (5.5.7) is interpreted here as the discarding of all the circles (for small V) except their upper arcs which span the contacts with the envelope. Thus between the envelope components we see a smooth family of curves of constant V (so labeled in the figure) which merge with the useable part of $\mathscr{C}$, which of course is on the locus of $V = 0$.

At about $V = 2$, we see that the envelope terminates. The circles burgeon from here on, and we only delete those arcs which would cover

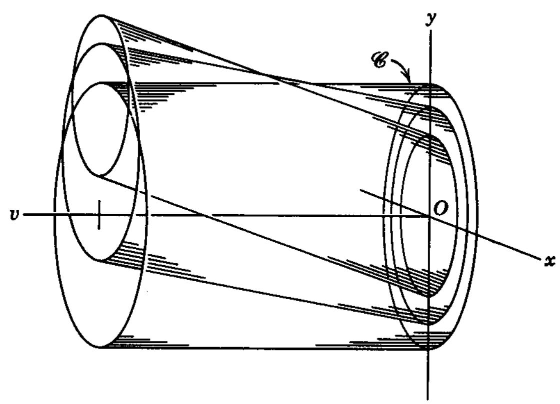


Figure 5.5.3

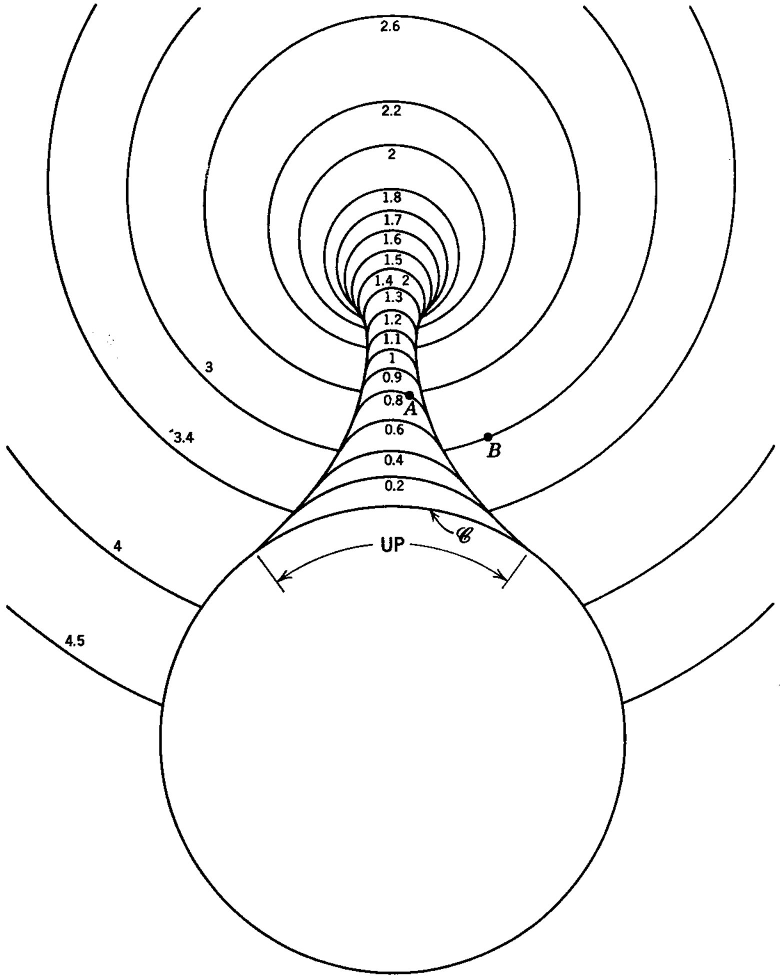


Figure 5.5.4

territory already labeled with smaller V. For all sufficiently large V (not shown) we retain the full circle.

The envelope is a section of a surface in two parts in $\mathscr{E}$ of a species that in Chapters 8 and 9 will be called barriers. Here the barrier is tangent to $\mathscr{C}$ at the boundary of the useable part. It is a surface that is never crossed during optimal play and marks discontinuities both in V and the optimal strategies. It delineates those starting positions leading to an optimal play which is a simple, direct chase from those involving a what we might, analogously to the homicidal chauffeur game, call a swerve.

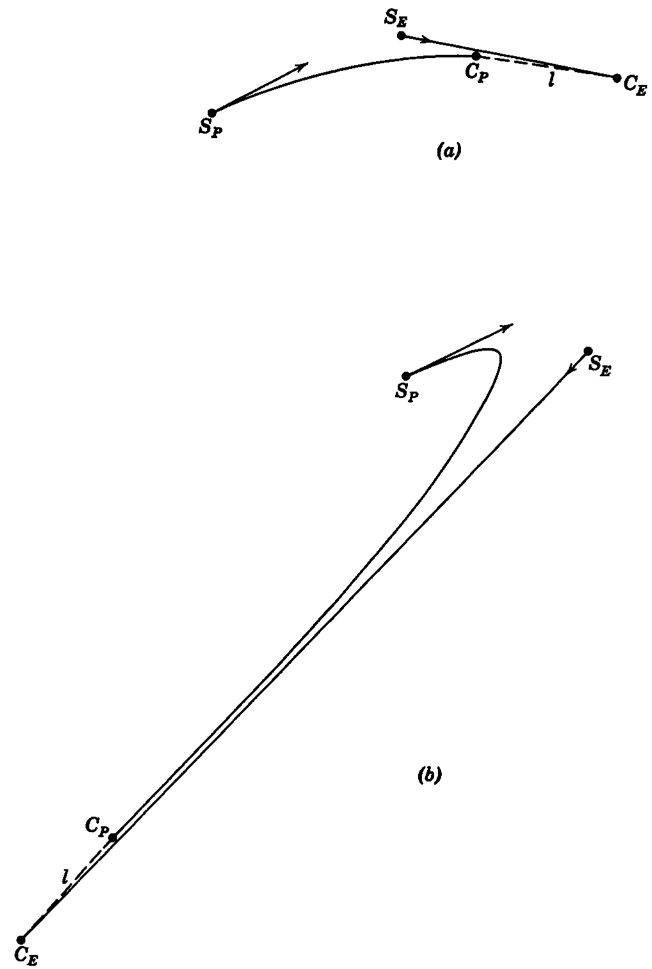


Figure 5.5.5

If the starting position is a point nestled between the barriers, such as A in Figure 5.5.4, the resulting play will be of the direct type; E will travel away from P with the latter dogging his heels. In this reduced space, for starting positions such as B outside the barrier, the resulting path in $\mathscr{E}$ will begin by first receding from $\mathscr{C}$, then skirting the barriers, and reaching $\mathscr{C}$ via the alley between them. "Physically" this means that the kinematics of P render him undeft enough to catch E directly; P cannot veer enough from his course to thwart E's sidestepping. Hence P must at first lower his speed so as to make a sufficiently sharp turn and then go after E, who has by now fled to a position roughly rear of P's starting point.

The realistic behavior of P and E are shown for these cases in Figure 5.5.5, (a) and (b). Here S_P, S_E denote the starting points of the players and C_P, C_E their positions at capture.

At b, E "swerves" to P's rear, penalizing the latter by forcing the sharp turn. P, an optimal player, anticipates the ploy; he points his thrust accordingly in a concordant direction (which is that of E's path). Even though the starting position at a is only slightly different, E could not get away with the same maneuver here; he would be caught abruptly if he headed tailward. Note that $V(A) = 0.8$, while $V(B) = 3.0$.

We can see now some significance in the assumption (5.5.9) that $Q > 0$. Were this not so, the "radius" of the cylinders of constant V would somewhere shrink to zero and the two barrier surfaces would intersect. We shall anticipate Section 9.3, where this question is more fully discussed, by remarking that, if the barrier is cut off beyond the intersection, the remaining surface appears to bound with $\mathscr{C}$ a portion of $\mathscr{E}$. For starting points within this portion the preceding analysis is valid. For external starting points, if E plays properly, P cannot capture at all.

Thus, as was remarked after (5.5.10), if P is to be able to capture from all starting positions, $F/k > w$ is a necesary condition. It is not sufficient, however. (See Exercise 9.3.1 or (9.3.7) if $k = 0$.)

5.6 AN OPTIMAL PROGRAM OF STEEL PRODUCTION

Here we demonstrate, by a simplified example, how our methods can be applied to certain production programs so as to attain the maximal yield. To fit our conceptions, the situation is best supposed "differential"; the discrete steps of actuality must be smoothed into continuity. What we obtain is a long-term view or a rounded-out picture.

A nation or other large enterprise embarks on a program of steel production. We will suppose, as is actually reasonable practice, that a certain amount of extant steel is to be an ingredient for the manufacture of additional steel. At any time the current supply of steel is to be allocated between this use, the manufacture of more steel mills, or the stockpile.

Our desideratum is to maximize the latter category at the elapse of a specified time T. How should we proceed? We might conjecture that at the outset, for example, all effort should be devoted to augmenting the number of mills; when this number is great enough we proceed at high blast to produce. When should the transition occur?

At any time t let M be the number of mills existing and S be the current supply of steel. Let ψ_M be the fraction of steel devoted to building more mills, ψ_S the fraction devoted to making more steel so that

$$\psi_M \geqslant 0, \quad \psi_S \geqslant 0, \quad \psi_M + \psi_S \leqslant 1.$$

Since $\psi_M S$ is the amount of steel allotted to mill building, we can write, supposing a linear relationship,

$$\dot{M} = c\psi_M S$$

for some $c > 0$. In a unit time let the quantity of new steel that can be made from a unit of the old be $a > 1$. Then the rate at which *new* steel is produced is $a\psi_S S$, but steel is consumed by doing so at the rate $\psi_S S$. Besides the steel allotted to mill building is an irrevocable deduction from the current supply and its rate is $\psi_M S$. Thus

$$\dot{S} = a\psi_S S - \psi_S S - \psi_M S.$$

A third state variable is T, the time until the end of the program. Therefore we have for the

KE:
$$\dot{M} = c\psi_M S.$$
$$\dot{S} = S[(a-1)\psi_S - \psi_M]$$
$$\dot{T} = -1.$$

The number of exant steel mills sets a bound to the output of new steel production. Therefore, for some $b > 0$, we must operate subject to the constraint

$$a\psi_S S \leqslant bM$$

or

$$\psi_S \leqslant \frac{bM}{aS}. \tag{5.6.1}$$

Thus in this problem the control variables will not always be subject to constant bounds.

For $\mathscr{E}$ we have the octant of 3-space: $M \geqslant 0$, $S \geqslant 0$, $T \geqslant 0$; $\mathscr{C}$ is the surface where $T = 0$, and we parameterize it by the partial initial conditions:

$$M = s_1 \geqslant 0$$
$$S = s_2 \geqslant 0$$
$$T = 0.$$

The payoff is terminal, for our goal is to maximize S when $T = 0$. Therefore

$$H = s_2.$$

The constraint (5.6.1) is effective only when $bM < aS$. If we defi[ne]

$$R = S - \left(\frac{b}{a}\right)M$$

we can say, only when $R > 0$. Accordingly we divide $\mathscr{E}$ into the two part[s] $\mathscr{E}_1$ where $R < 0$, and $\mathscr{E}_2$ where $R > 0$, by the quarter-plane $\mathscr{R}$ on whic[h] $R = 0$. In $\mathscr{E}_1$, there are ample extant mills to handle all the steel, but i[n] $\mathscr{E}_2$ the upper limit on production rather than a supply shortage is incurre[d] by a mill. Figure 5.6.1a depicts $\mathscr{E}$; note that the vectograms have tri-

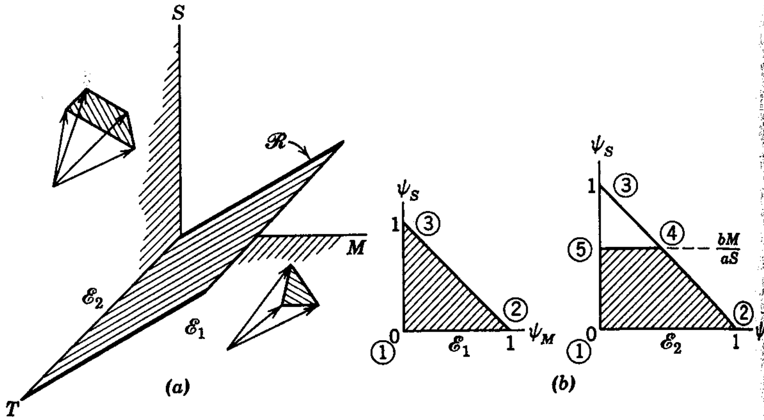


Figure 5.6.1

angular headplanes in $\mathscr{E}_1$ and trapezoidal ones in $\mathscr{E}_2$. Such are engendere[d] by the constraints on the control variables which limit them to the shade[d] regions shown at (b).

The ME_1 is

$$\max_{\psi} \{cSV_M\psi_M + SV_S[(a-1)\psi_S - \psi_M]\} - V_T = 0.$$

To find $\bar{\psi}$ we observe that, in either $\mathscr{E}_1$ or $\mathscr{E}_2$, we are maximizing a linea[r] function of ψ_M, ψ_S over a convex polygon. The maximum will alway[s] occur at a vertex. In Figure 5.6.2 we have written the value of the brace o[f] the ME, at the appropriate places. The vertices will be denoted by th[e] numerals encirlced in the previous figure, and the corresponding values o[f] the brace by m_j ($j = 1, \ldots, 5$). We will also refer to the local strategies b[y]

these numbers, for example, at 4, $\psi_M = 1 - bM/aS$, $\psi_S = bM/aS$. Of course, 1, 2, 3, apply in $\mathscr{E}_1$ and 1, 2, 4, 5, in $\mathscr{E}_2$.

When we derive the RPE, the nonconstant bounds on ψ necessitate a modification of our standard procedure. The change is simple; when differentiating the ME with respect to a control variable, we include its appearance as an argument of $\bar{\psi}_M$ or $\bar{\psi}_S$. Inasmuch as the ME_2 will always be one of

$$m_j - V_T = 0, \qquad j = 1, \ldots, 5 \tag{5.6.2}$$

the RPE are

$$\overset{\circ}{M} = -cS\bar{\psi}_M \qquad \overset{\circ}{V}_M = \partial m_j/\partial M$$
$$\overset{\circ}{S} = -S[(a-1)\bar{\psi}_S - \bar{\psi}_M], \qquad \overset{\circ}{V}_S = \partial m_j/\partial S$$
$$\overset{\circ}{T} = 1 \qquad \overset{\circ}{V}_T = 0.$$

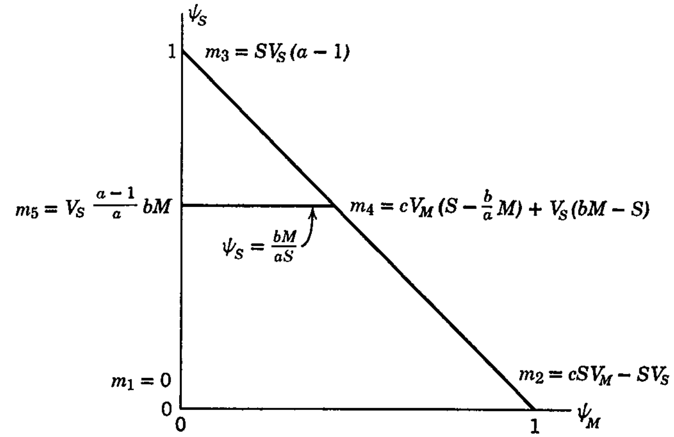


Figure 5.6.2

We make some general observations.

The value of the maximizing m_j is constant over all of any optimal path. (5.6.3)

For when V is smooth, (5.6.2) and the final KE show that $\overset{\circ}{m}_j = 0$. At a transition surface one m_j will supplant another, but they must have a common value at the junction. At a crossing of $\mathscr{R}$, m_3, m_4, m_5 coalesce and so are numerically equal.

Always

$$V_S \geqslant 1, \qquad V_M \geqslant 0. \tag{5.6.4}$$

For suppose, starting from a point (M, S, T) of $\mathscr{E}$, we play optimally. If the steel were initially augmented by an amount S_1, we could ignore the increment and use the former strategy. Thus we could get a payoff at least equal to the old Value + S_1. The second inequality follows similarly if we ignore an increment of M. We can certainly use the old $\bar{\varphi}_M$, $\bar{\varphi}_S$, the constraint (5.6.1) being now the weaker.

Never is m_1 optimal.[14] (5.6.5)

For, from (5.6.4), it is always dominated by m_3 or m_5.

If we put

$$Q = cV_M - V_S$$

then

$$m_4 - m_5 = QR \tag{5.6.6}$$

and

$$m_2 = QS \tag{5.6.7}$$

The m_j of Figure 5.6.2 and a simple computation give these results.

The initial conditions are completed in the usual way; on $\mathscr{C}$

$$V_M = \frac{\partial H}{\partial s_1} = 0, \quad V_S = \frac{\partial H}{\partial s_2} = 1.$$

We turn now to the solution in $\mathscr{E}_2$. As on $\mathscr{C}$, $Q = -1$, $R > 0$, we have, from (5.6.6)

$$m_5 > m_4$$

and from (5.6.7)

$$m_2 < 0.$$

As $m_5 > 0$, the max m_j at $\mathscr{C}$ is m_5, that is, $\bar{\varphi}_S = bM/aS$, $\bar{\varphi}_M = 0$. All steel and no mill production appears reasonable near the end of the program.

The RPE are here

$$\begin{aligned} \overset{\circ}{M} &= 0, & \overset{\circ}{V}_M &= kV_S \quad \left(= \frac{\partial m_5}{\partial M}\right) \\ \overset{\circ}{S} &= -kM, & \overset{\circ}{V}_S &= 0 \\ \overset{\circ}{T} &= 1, & \overset{\circ}{V}_T &= 0 \end{aligned}$$

where we have written, for short,

$$k = b\,\frac{a-1}{a}.$$

The integrals are

$$\begin{aligned} M &= s_1 & V_M &= k\tau \\ S &= s_2 - ks_1\tau, & V_S &= 1 \\ T &= \tau & V_T &= m_5 = ks_1. \end{aligned}$$

[14] It will turn out that neither is m_2 ever optimal. Is there a way of establishing this fact at this stage?

Consequently,

$$\begin{aligned} Q &= ck\tau - 1 \\ R &= s_2 - \left(\frac{b}{a}\right)s_1 - s_1k\tau. \end{aligned}$$

When $\tau = \tau_1 = s_2/s_1k - b/ak$, R vanishes; the path leaves $\mathscr{E}_2$. When $\tau = \tau_2 = 1/ck$, Q vanishes; we should expect, by (5.6.6), the dominance of m_5 to be yielded to m_4. On any path one or the other occurs, depending on the relative magnitudes of τ_1 and τ_2, which depend on s_2/s_1. It is not hard to show that the two classes of paths together fill all of $\mathscr{E}_2$ with $T \leqslant \tau_2$ and intersect all of points of $\mathscr{R}$ subject to this inequality. The surface, which we shall soon see to be a transition one, consisting of those points of $\mathscr{E}_2$ for which

$$T = \tau_2 = \frac{a}{bc(a-1)} \tag{5.6.8}$$

similarly is met at all points.

It is clear that m_2 is out of the running, for from (5.6.7), $m_2 < 0$ as long as $S > 0$ and $\tau < \tau_2$.

Now let us turn to paths on the far side of the TS. As initial conditions, we use (the s_i are fresh parameters)

$$\begin{aligned} M &= s_1, & V_M &= k\tau_2 = \frac{1}{c} \\ S &= s_2, & V_S &= 1 \\ T &= \tau_2, & V_T &= ks_1. \end{aligned}$$

Taking 4 for strategies, we now have the RPE:

$$\begin{aligned} \overset{\circ}{M} &= -cS\left(1 - \frac{bM}{aS}\right) = -cR & \overset{\circ}{V}_M &= b\left(-\frac{c}{a}V_M + V_S\right) \\ \overset{\circ}{S} &= -bM + S & \overset{\circ}{V}_S &= cV_M - V_S = Q \\ \overset{\circ}{T} &= 1 & \overset{\circ}{V}_T &= 0. \end{aligned}$$

We shall obtain full results pertaining to the optimal strategies without integrating these equations, although to find V we may have to do so.

As long as a path remains in $\mathscr{E}_2$, R must be positive. We shall show that Q, which equals zero for $\tau = 0$, is otherwise positive. From (5.6.6), such will imply that $m_4 > m_5$.

We have

$$\begin{aligned} \overset{\circ}{Q} &= bc\left(-\frac{c}{a}V_M - V_S\right) - Q \\ &= -k_1Q + ckV_S \end{aligned}$$

where $k_1 = 1 + bc/a > 0$. The elementary formula for the integral of such a differential equation, with $Q(0) = 0$, gives us

$$Q = cke^{-k_1\tau}\int_0^\tau e^{+k_1 u}V_S(u)\,du$$

and so, from (5.6.4), our result follows, as exponentials are always positive.

We now show that m_2 cannot dominate; it follows that m_4's sway is permanent. From Figure 5.6.2 and the RPE

$$m_4 - m_2 = -\frac{cb}{a}MV_M + MV_S = M\mathring{V}_M.$$

In $\mathscr{E}$, $M \geqslant 0$ and on the TS, as $a > 1$,

$$\mathring{V}_M = b\left(-\frac{1}{a} + 1\right) > 0.$$

We assert that $\mathring{V}_M$ remains positive. If it did not, let τ_0 be the lowest τ for which $\mathring{V}_M = 0$. From the RPE, at $\tau = \tau_0$,

$$\mathring{\mathring{V}}_M = b\left(-\frac{c}{a}\mathring{V}_M + \mathring{V}_s\right) = b\mathring{V}_s = bQ > 0.$$

But it is absurd that the derivative of $\mathring{V}_M$ be positive at its lowest zero.

Finally we shall show that this class of optimal paths completely fills the part of $\mathscr{E}_2$ beyond the transition surface. Let (M, S, T) be a point of this set, so that $R = S - (b/a)M > 0$. The *progressive* path from this point will satisfy (reverse the signs in the RPE)

$$\dot{M} = cR$$
$$\dot{S} = bM - S$$
$$\dot{T} = -1$$

and so

$$\dot{R} = bM - S - \frac{bc}{a}R.$$

If at some later time R equals zero, let t_0 be the first such point. At t_0, $bM = aS$ and

$$\dot{R} = bM - S = (a - 1)S > 0.$$

and we have the same type of absurdity as before. Let us note in passing, had we started from a point of R (with strategy 4), this same reasoning shows that $\dot{R} > 0$ and so the path would (progressively) enter $\mathscr{E}_2$.

Thus $R > 0$. As $\dot{M} > 0$, also $M > 0$ and $S = R + (b/a)M > 0$. Thus the path remains in $\mathscr{E}_2$. Ultimately it must reach the transition surface and so be the reverse of one of our retrograde paths. Thus $\mathscr{E}_2$ beyond the TS is filled; besides a path goes through each point of $\mathscr{R}$.

Finally, we treat $\mathscr{E}_1$. We shall show that everywhere here

$$m_3 > m_2 \tag{5.6.9}$$

and, in view of (5.6.5), the optimal strategy is fixed at 3.

On $\mathscr{C}$, from Figure 5.6.2 and the initial conditions,

$$m_3 = s_2(a - 1), \qquad m_2 = -s_2$$

and so (5.6.9) holds. It also does on $\mathscr{R}$ for each point there is met by a path from $\mathscr{E}_2$ on which m_4 or m_5 dominates and these agree with m_3 on $\mathscr{R}$.

We now show that if (5.6.9) holds at the initial point of a path, it does so throughout. The RPE here are ($\bar{\psi}_S = 1$, $\bar{\psi}_M = 0$)

$$\mathring{M} = 0, \qquad \mathring{V}_M = 0$$
$$\mathring{S} = -S(a - 1), \qquad \mathring{V}_S = V_S(a - 1)$$
$$\mathring{T} = 1, \qquad \mathring{V}_T = 0$$

with the general integrals (the subscript 0 denotes an initial value)

$$M = M_0, \qquad V_M = V_{M0}$$
$$S = S_0e^{-(a-1)\tau}, \qquad V_S = V_{S0}e^{(a-1)\tau}. \tag{5.6.10}$$

Then, for $\tau > 0$ (or τ_0)

$$m_2 = S(cV_M - V_S) = S_0e^{-(a-1)\tau}(cV_{M0} - V_{S0}e^{(a-1)\tau})$$
$$= S_0(cV_{M0}e^{-(a-1)\tau} - V_{S0}) < m_{20}.$$

As m_2 remains blow its initial value, and m_3, by (5.6.3), remains at its, our result follows.

Finally, it is easy to see that the retrograde paths of (5.6.10) (left), emanating from $\mathscr{C}$ and $\mathscr{R}$, will fill $\mathscr{E}_1$ completely.

The complete optimal strategies are mapped in $\mathscr{E}$ in Figure 5.6.3. In $\mathscr{E}_1$, where there are more than enough mills, we should, as we have just learned, devote all steel to steelmaking. But one typical possibility is shown if we start at the point X of $\mathscr{E}_1$. We make steel exclusively until we have enough to saturate the existing mills. Then X is at A on $\mathscr{R}$. From A to B the mills are used at their full capacity and the excess ingredient steel is used to build more mills. At B, time $a/bc(a - 1)$ from termination, we continue capacity manufacture but stop mill building and stockpile the excess. We so continue until C, which lies on $\mathscr{C}$.

As our description of the optimal strategy is complete, and of the paths at least qualitatively so, the Value can be computed by routine integrations.

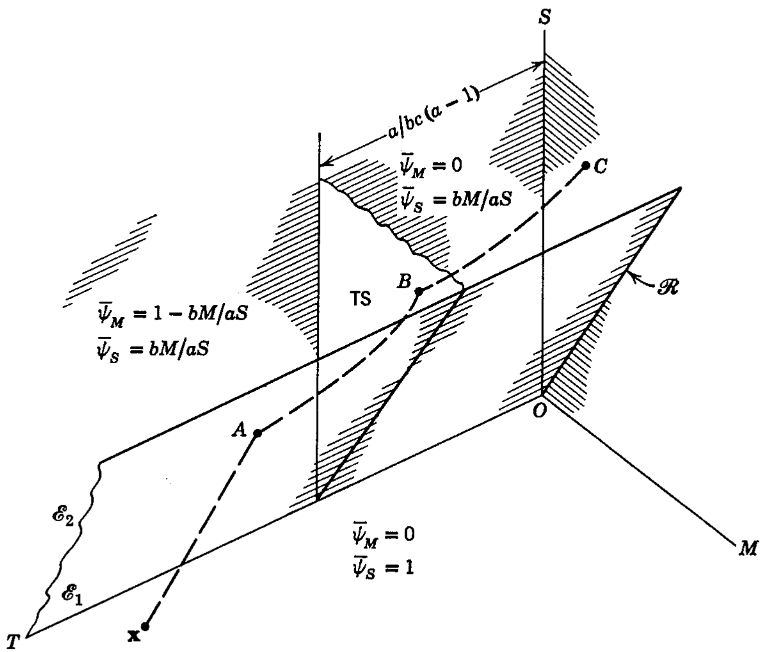


Figure 5.6.3

5.7. INTEGRAL CONSTRAINTS

The contents of this section will apply to a problem in the Appendix but will not be used in the ensuing text itself.

In the classical calculus of variations there are problems where the paths are restricted to those which render prescribed constant values to one or more given integrals. The relevant generalization here is

$$\int L(\mathbf{x}, \phi, \psi)\, dt = C \tag{5.7.1}$$

the integral extending over the path from the starting point to $\mathscr{C}$. We will deal with a single constraint of the type (5.7.1); the generalization to more is obvious.

We introduce a new state variable and adjoin to the kinematic equations

$$\dot{x}_{n+1} = -L(\mathbf{x}, \phi, \psi). \tag{5.7.2}$$

The new $\mathscr{C}$ is the same as the old with $x_{n+1} = 0$. Following our usual procedure we discard all starting points but those lying in the plane: $x_{n+1} = C$. The condition (5.7.1) must hold and our general technique

ensures the minimax with respect to ϕ- and ψ-strategies under this constraint. Often the above new $\mathscr{C}$ will be deficient in dimension. We then can proceed in our usual way (Section 2.3): surround the terminal curve by neighborhood of radius δ, use the boundary as $\mathscr{C}$ and take the limit as $\delta \to 0$.

But an alternative procedure seems more logical. The purpose of a δ-neighboring surface is to supply initial values of the V_i. We can do so directly by taking

$$V_{n+1} = -\lambda \tag{5.7.3}$$

where λ is a parameter that can label the family of paths emanating from $\mathscr{C}$ (which is of dimension $n - 1$ in an $(n + 1)$-space and so should normally not seat an n parameter family of paths). Together with the usual $n - 1$ initial conditions (on $\mathscr{C}$)

$$V_{s_j} = \sum_i V_i \frac{\partial h_i}{\partial s_j}, \qquad j = 1, \ldots, n - 1$$

and the ME_2, we have $n + 1$ equations to be solved for $V_1, \ldots, V_{n+1}$ in terms of $s_1, \ldots, s_{n-1}, \lambda$.

The only KE in the augmented set in which x_{n+1} appears is (5.7.2). Hence one of the RPE will be

$$\mathring{V}_{n+1} = 0. \tag{5.7.4}$$

It follows that λ will be constant over each optimal path, retaining the value it had on $\mathscr{C}$.

The advantage of all this is that there are significant instances when we are interested in the restriction (5.7.1) only superficially; what really concerns us is

$$\int L(x, \phi, \psi)\, dt \leqslant C. \tag{5.7.5}$$

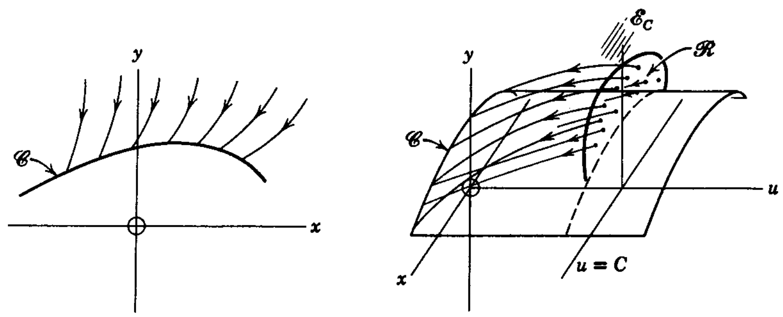
In practical problems such happens when there is a resource (fuel, time, etc) x_{n+1} with a limit which one of the players must not exceed. Usually we can expect that optimal play demands that he exhaust all the resource allotted him; then a condition (5.7.1) suffices to express the constraint, although (5.7.5) is the literally true criterion.

But there are cases where too much expenditure of the resource, even though allowed, is deleterious. Such is manifested from the point of view, say, of the minimizing player P, at a state (point of $\mathscr{E}$) by

$$\frac{\partial V}{\partial x_{n+1}} > 0$$

holding there. In view of (5.7.3) and (5.7.4) this condition means that

$$\lambda < 0$$



(a) (b)

Figure 5.7.1

at the relevant point of $\mathscr{E}$. Hence we attain the optimal expenditure of resource by the restriction on the initial conditions of

$$\lambda \geqslant 0. \tag{5.7.6}$$

Of course, when the integral may $\geqslant C$, but not fall below, we work only with $\lambda \leqslant 0$.

Let us make this all clear by a simple yet general graphic illustration. With $n = 2$, take KE of the usual form $\dot{x} = f$, $\dot{y} = g$, so that the ME_2 is

$$V_x\bar{f} + V_y\bar{g} + \bar{G} = 0 \tag{5.7.7}$$

where the bars indicate that the usual $\bar{\phi}$ and $\bar{\psi}$ appear as arguments. In Figure 5.7.1a, $\mathscr{C}$, specified by $x = X(s)$, $y = Y(s)$ and on it $V = H(s)$, and the optimal paths are shown, assuming the solution to have been attained in our customary way. The part of the plane above $\mathscr{C}$ is $\mathscr{E}$.

Now let us consider the new game which arises when we adjoin the strict equality constraint (5.7.1). To the KE is added

$$\dot{u} = -L$$

and the new ME_2 is now

$$V_x\bar{f} + V_y\bar{g} + \bar{G} - \bar{L}V_u = 0 \tag{5.7.8}$$

and we should note the $\bar{\phi}$ and $\bar{\psi}$, functions now of x, y, V_x, V_y, V_u, are not necessarily the same as in (5.7.7) and thus neither are $\bar{f}, \bar{g}, \bar{G}$.

Figure 5.7.1b shows the enlarged $\mathscr{E}$. Note that the new $\mathscr{C}$ is in the $u = 0$ plane and is otherwise the same as the old:

$$x = X(s), \quad y = Y(s), \quad u = 0. \tag{5.7.9}$$

The new $\mathscr{E}$ consists of the points above the surface which is the u-translation of $\mathscr{C}$, thus of x, y, u such that $(x, y) \in$ old $\mathscr{E}$, $u \geqslant 0$.

The set $\mathscr{E}_C$ defined by $x, y \in$ old $\mathscr{E}$, $u = C$, is comprised of starting points such that the constraint (5.7.1) holds.

For initial conditions we now use first the values (5.7.9) of x, y, u and the values of the V_i obtained by solving simultaneously

$$X'(s)V_x + Y'(s)V_y = H_s \text{ (as in the case of no constraint)}$$
$$V_u = -\lambda$$

and the new ME (5.7.8) with the values (5.7.9) as arguments in $\bar{f}, \bar{g}$ etc.

From the RPE—the old ones with $\mathring{u} = L$ and $\mathring{V}_u = 0$ added—with the above initial conditions we obtain a two-parameter (s and λ) family of paths emanating from $\mathscr{C}$. That is, from each point of $\mathscr{C}$ there is a family of paths, one for each λ. We may expect that the paths will meet $\mathscr{E}_C$ in a subset $\mathscr{R}$ of it as indicated in (b) of the figure.

Suppose we wish the solution of the constrained problem with a specified starting point x^0, y^0. Then we find the corresponding point (x^0, y^0, C) of $\mathscr{E}_C$. If it lies in $\mathscr{R}$, then the path through it will be the optimal one for the augmented game; and the projection of this path on the plane: $u = 0$ will be the optimal one in the original space.

If the point (x^0, y^0, C) lies outside of $\mathscr{R}$, then there is no path, which means, in the original terms, that it is impossible to get from the starting point to $\mathscr{C}$ without violating the constraint. In practice, the boundary of $\mathscr{R}$, which delineates those impossible cases, may usually be identified as the limiting curve in $\mathscr{E}_C$ where $\lambda = \infty$.

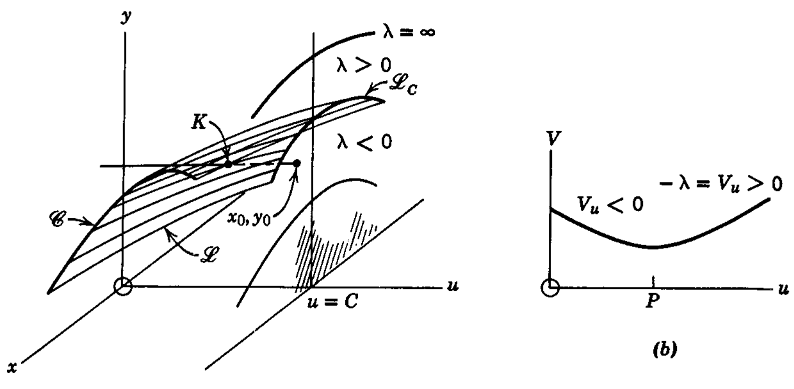
Note that on starting points of the curve in $\mathscr{E}_C$ for which $\lambda = 0$, optimal play is the same as in the original unconstrained game. For if $V_u = \lambda = 0$, it is clear that formal calculations of the original game are unchanged when we adjoin the constraint. The paths of (b) in the figure for which $\lambda = 0$, when projected on the $u = 0$ plane, are the paths of (a).

Now let us turn to the case of the one-sided constraint (5.7.5). We shall assume that, in an effective two-player game, the constraint integral is governed by one player alone. For otherwise we might expect such phenomena as the opponent striving to violate the constraint; for the game to retain sense, he should reap a bonus in payoff if he succeeds and such appears to require a basic reformulation of the original rules. Certainly our assumption holds in many cases derived from reality. For example, if one of the players controls a moving craft, the constraint integral might express a range or fuel limitation and it would be this player alone who regulates its expenditure. For definiteness we shall take P, the minimizing player, as enjoying this control.

The admissible starting points are now the set Q defined

$$(x, y) \in \mathscr{E}, \qquad 0 \leqslant u \leqslant C$$

for such and only such ensure that $\int L \leqslant C$.



(b)

(a)

Figure 5.7.2

Now Q will be cut by the surface $\mathscr{L}$ on which $\lambda = 0$, as shown in Figure 5.7.2a, which meets the plane where $u = 0$ at $\mathscr{C}$ and meets $\mathscr{E}_C$ at a curve we shall call $\mathscr{L}_C$. We shall assume an orientation which, in our figure, reads: $\lambda < 0$ below $\mathscr{L}$ and $\lambda > 0$ above $\mathscr{L}$.[15]

Suppose now we are assigned a starting point x^0, y^0 in the original problem. In the constrained problem we are at liberty to use any starting point on the horizontal line through Q: $x = x^0$, $y = y^0$, $0 \leqslant u \leqslant C$. The optimal will be that one where V is least. Suppose first the point x^0, y^0 in $\mathscr{E}_C$ lies below $\mathscr{L}_C$. Then the horizontal line through it will meet $\mathscr{L}$ at a point K. As we know, from the KE, $\mathring{V}_u = 0$, and the initial condition, $V_u = -\lambda$, that $V_u = \partial V/\partial u = -\lambda$ throughout Q, V will vary along the line as at (b) of the figure. For to the right of K, $V_u > 0$, at K, $V_u = 0$, etc. Thus the minimum occurs at K; it should be the starting point under optimal play. As

$$\int L\, dt = u \text{ (at } K) < C,$$

the optimal path should be the same as if there were no constraint, and P may ignore it.

On the other hand, if x^0, y^0 (on $\mathscr{E}_C$) lies above $\mathscr{L}_C$, then on the horizontal line,

$$V_u = \frac{\partial V}{\partial u} = -\lambda < 0$$

[15] We shall not stop to investigate the validity or alternatives to this assumption. It would appear to hold in most realistic cases.

and the best starting point has $u = C$. For x_0, y_0 on $\mathscr{L}_C$, of course, best play is along the optimal path where $\lambda = 0$ and the constraint if fulfilled exactly but without sacrifice. Thus we can conclude:

> *In the augmented game, when $\lambda \leqslant 0$ at the starting point, optimal play is the same as in the original unconstrained game but it must be altered when $\lambda > 0$.*

Note that if the maximizing player controls the integral, this statement is still true if the adjoined KE is taken as $V_u = +\lambda$.

Research Problem 6.3.1. What are the formal conditions that dictate that $\int L\, dt$ is under the control only of a particular player.

For a pristine example we revert to the simple classical problems of the isoperimetric type. They concern curves of specified length bounding, or partially bounding, with diverse end conditions, the maximal area. The textbook standby has the curve joining two given points in the upper halfplane; the area beneath, in the sense of elementary calculus, is to be maximized.

Example 5.7.1. A classical isoperimetric problem. To translate to our terms let the curve be described by a point moving with unit speed. Thus we write the
KE

$$\dot{x} = \cos\psi$$
$$\dot{y} = \sin\psi.$$

As the area beneath is $\int y\dot{x}\, dt$ we take

$$G = y\cos\psi.$$

Because of the unit speed, arc length is the same as time and so to form (5.7.1) we take

$$L = 1.$$

Thus the additional KE is

$$\dot{T} = -1.$$

We are led to the ME$_1$

$$\max_{\psi}\,[(V_x + y)\cos\psi + V_y\sin\psi] - V_T = 0.$$

Putting

$$\rho = \sqrt{(V_x + y)^2 + V_y^2}$$

we have

$$\cos\bar{\psi} = \frac{V_x + y}{\rho}, \qquad \sin\bar{\psi} = \frac{V_y}{\rho}$$

and the ME_2 is

$$\rho - V_T = 0.$$

The RPE:

$$\overset{\circ}{x} = \frac{-(V_x + y)}{\rho} \qquad \overset{\circ}{V}_x = 0$$
$$\overset{\circ}{y} = \frac{-V_y}{\rho} \qquad \overset{\circ}{V}_y = \frac{V_x + y}{\rho}$$
$$\overset{\circ}{T} = 1 \qquad \overset{\circ}{V}_T = 0.$$

We can see at once that any integral path of this system is an arc of a circle. For V_x and V_T are constant; from the ME_2, so is ρ. The linear subsystem of differential equations for y and V_y have the familiar integrals

$$y + V_x = C_1 \cos\left(\frac{\tau}{\rho} + C_2\right)$$
$$V_y = C_1 \sin\left(\frac{\tau}{\rho} + C_2\right). \qquad (5.7.10)$$

Finally, we note from the RPE that x differs from $-V_y$ by a constant.

To illustrate the initial conditions, let us take as the original $\mathscr{C}$ the line $\mathscr{L}$ where $x = X > 0$ and starting points with $x \leqslant X, y \geqslant 0, T > X - x$. We wish to find the curve of specified length T extending from this (x, y) to some point of $\mathscr{L}$ which subtends the maximal area beneath it.

The final $\mathscr{C}$ is

$$x = X, \quad y = s(\geqslant 0), \quad T = 0.$$

On it V_s (as $V = 0$ on $\mathscr{L}$) $= 0 = V_y$ and

$$V_T = \lambda.$$

In virtue of our discussion we limit ourselves to

$$\lambda \geqslant 0.$$

Then also on $\mathscr{C}$, using the ME_2,

$$\rho = |V_x + s| = \lambda.$$

As the nature of the paths dictate that $\overset{\circ}{x}(0) \leqslant 0$, the first RPE shows that on $\mathscr{C}$, $V_x + y \geqslant 0$. Then the initial conditions are completed by

$$V_x = \lambda - s.$$

Tailoring the integrated RPE to fit these, we have

$$x = X - \lambda \sin\frac{\tau}{\lambda}$$
$$y = s - \lambda\left(1 - \cos\frac{\tau}{\lambda}\right) \qquad (5.7.11)$$
$$T = \tau.$$

Clearly our curve is the arc of a circle with radius λ and center on $\mathscr{L}$ with ordinate $s - \lambda$. Obviously if T is within reasonable limits, there will be just one such arc of length T, ending at (x, y) and meeting $\mathscr{L}$ at a point above its center. This is the well-known classical answer.

We make the obvious observation that from the ME_2

$$V_T = \lambda = \rho \geqslant 0$$

so that we never lose area because of too much length.

If T is large enough, we should realize that there is nothing in our formulation, prohibiting the point from making more that one circuit of his circular optimal path. Area is of course reckoned with sign as in elementary calculus.

Finally, let us study the limiting case as $\lambda \to \infty$. It is easy to see that in the limit (5.7.11) becomes

$$x = X - \tau$$
$$y = s$$
$$T = \tau$$

corresponding to starting points whose distance from $\mathscr{C}$ is just T, so that there is no choice of paths. These cases delineate the possible.

Exercise 5.7.1. Solve the typical classical isoperimetric problem where the sought curve is to connect two given endpoints in the upper half-plane. (Surround one of the endpoints, say P_1, by a circle of small radius δ to obtain $\mathscr{C}$, etc.)

Problem 5.7.1. Make a two-person game of the last exercise by allowing a second player P to move P_1 with simple motion at a speed <1. His objective will be to minimize the area under the curve. Its length is still to be T, but P can be considered victorious if he can render such impossible to the original player E.

It is hard to mine anything new from such well dug ground as these isoperimetric problems. But the problem below is novel in that it asks for fewer end conditions than its classic brethren. Yet it is meaningful and fits our ideas nicely.

Problem 5.7.2. What curve (or curves) maximize the area beneath when the only constraints are

1. The length is a given $T > 0$.
2. *One* endpoint is given?

Research Problem 5.7.1. Show that, up to an arbitrary multiplicative constant (perhaps ± 1), our λ is the same as the Lagrange multiplier in the classical treatment.

CHAPTER 6

Efferent or Dispersal Surfaces

The rich variety of singular surfaces which can be entailed in the solutions to differential games is the key to the important, often predominant, phenomena which fall outside the scope of mere differential equations. This chapter is the first of several which emphasize a particular type. Dispersal surfaces, although simple in principle, are often the seat of mixed strategies and conflicting decisions by the participants. There has been a great deal of misunderstanding about such situations, which we endeavor to clarify in the central sections.

The first section contains a classification scheme for singular surfaces and some general observations on their roles. Dispersal surfaces are defined, singled out, and illustrated by examples.

Section 6.7 is devoted to the geometric method of solving certain pursuit games. Here too are the above difficulties—special positions which at first appear to involve an unresolvable decision balance.

The final sections contain new examples, some presented as problems.

6.1. SINGULAR SURFACES

As has been mentioned earlier, there are frequently two aspects or stages to the solution of a problem of differential games. One, termed the *in the small*, concerns the integration of the RPE, and in the examples of the previous chapters this procedure bulked large. The other, termed *in the large*, consists of ascertaining certain *singular surfaces* which generally separate regions of different behaviors of the integrals of the RPE.

A singular surface is an $(n-1)$-dimensional manifold in $\mathscr{E}$ on which regular behavior of the solution, as an integral of the ME, fails.

To classify singular surfaces, we can consider the optimal paths on the two sides, for we assume, at least locally, that a surface, interior to $\mathscr{E}$ separates it into two regions, locally "sides." There are four obvious possibilities for the path behavior on each, which are, with their notations:

The paths enter the surface. (+)
They leave it. (−)
They do neither, that is, are parallel to the surface when sufficiently nearby. (p)
There are no paths. (0)

As each condition may hold on either side there are 16 possibilities, which we designate by such symbols as $(-, p)$. Further, the surface itself may be comprised of a set of paths; we will denote this event by (u). For example, the singular surface of Figure 6.1.1b is of type $(+, u, +)$.

This classification thus comprises 32 possibilities. Not all of them need be realized in practice, for example, the cul-de-sac $(+, +)$. Transition surfaces, where a control variable abruptly changes in value, have appeared in examples of the preceding chapter. They are of type $(+, -)$, but so is an ordinary surface in $\mathscr{E}$ if it cuts an optimal path at each of its points. (But an ordinary surface, if a union of paths, is (p, u, p).)

This classification is exhaustive geometrically, that is, in terms of the configurations of optimal paths. As such it suffices for most of our purposes, but full consummation would demand in some cases a finer subdivision in terms of the local optimal strategies. For example, from each point of a singular surface of type $(-, u, -)$ three paths emanate. Optimal play could require $\mathbf{x}$ to stay on the surface (the (u)) when there, but take a branch path (the proper $(-)$) when slightly away. Or all three choices are optimal. Or a mixed strategy (such is not impossible, as we shall shortly see) is best, etc. Further a (0) may signify that no solution exists or many do with all paths optimal.

But mere taxonomy in itself does not constitute a theoretical structure. We have as yet discovered no unified theory based on the above classification. On the contrary, the ideas surrounding the various kinds of singular surfaces seem extremely disparate. Much of the ensuing text will be devoted to particular types.

Differential games appears to have an extraordinary furcate structure. No matter to what extent the theory is seemingly mastered, new and perplexing phenomena crop up, even in the most innocuous looking cases. We shall drop a sample of such in the reader's lap in Section 6.10.

As we successively master each novel quandary, some type of singular surface usually is the key. For they delineate, as boundaries, the regions of special phenomena or they act as the bearers of initial conditions which

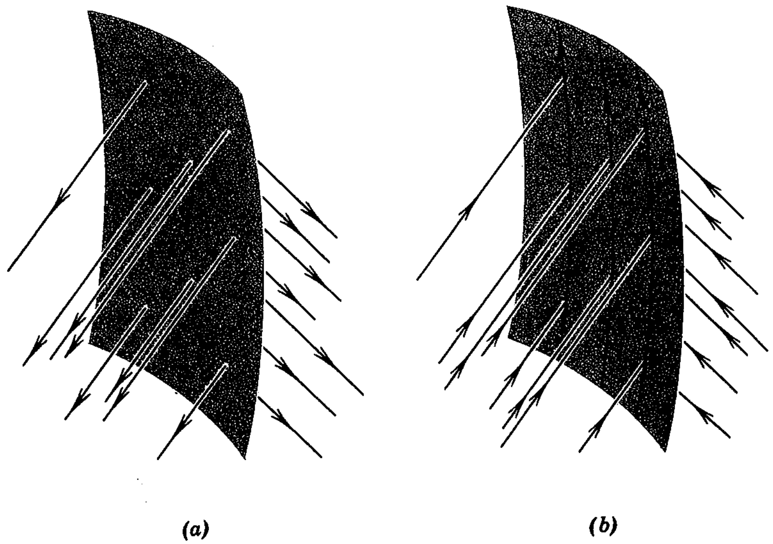


Figure 6.1.1

generate new families of paths. Thus it is that a theory of singular surfaces based on their types will be recondite, perhaps nonexistent, and it is thus that we devote so much effort to particular types.

In this chapter we shall study the type $(-, -)$, depicted, for $n = 3$, in Figure 6.1.1a. At (b) is a specimen of type $(+, u, +)$, such as will be studied in the next chapter, which with its tributary paths, may have the same geometric configuration. Despite their pictorial similarity, these two singular surfaces signify very distinct phenomena.

We shall term the $(-, -)$ case a *dispersal* or *efferent* surface, and the $(+, u, +)$ a *universal or afferent* surface.[1] As we shall soon see, the former entails a quandary: one or both players must decide by which of the two routes he shall leave the surface. On the other hand, a universal surface is an assembly of especially advantageous routes to the player controlling the creation of such a surface and the tributary paths generally represent his preliminary efforts to reach this desirable highway.

6.2. DISPERSAL SURFACES

A rudimentary but typical instance occurs in a game of tag with a large obstacle—say, circular—in the playing region. We assume the players

[1] The latter choices of names are of course borrowed from the neural nomenclature. However, the former choices are more descriptive of the roles of the surfaces and we prefer them.

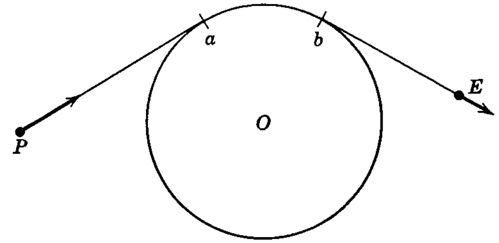


Figure 6.2.1

have simple motion, with P's speed greater than that of E. If time of capture is the payoff, it requires no sophisticated analysis to realize that the course of an optimally played partie will often be as sketched Figure 6.2.1. The tangent to the circle (at b) through his initial position will be E's best route, while P will travel first along the tangent Pa to a, then will traverse the arc ab until hot on E's trail bE, remaining on this line until capture.

But suppose that the starting positions both lie on a line through the center O of the circle with the latter between the players. The symmetry besets each player with a quandary; each has two equally good tangents to use as routes. The set of all such symmetric positions constitutes our simple but typical dispersal surface.[2]

Another instance of a dispersal surface is in the homicidal chauffeur game. Should E be directly and sufficiently far rearward of P, each player will have two optimal strategies; P, for example, must choose between a sharpest left and right turn.

6.3. THE NATURE OF DISPERSAL SURFACES

We are assuming as part of the definition of a dispersal surface that strategies exist so that either of the two optimal paths emanating from any point of it, may be utilized. As a consequence, it is clear that the traversal of either of a pair of paths from the same point of the surface will lead to the same payoff. For this payoff will be the Value of the game starting from this point.

In a one-player game, consequently, the player may choose either path indifferently, both being optimal. But with two players, there may be a characteristic dilemma: the choice of each player will depend on that of his opponent. Generally one player will desire matching choices; the other the opposite, so that at the instant they are doing something like playing

[2] Clearly a reduced space of three dimensions suffices: say, the distances OP and OE and the angle POE. The set where the latter is π must contain the dispersal surface.

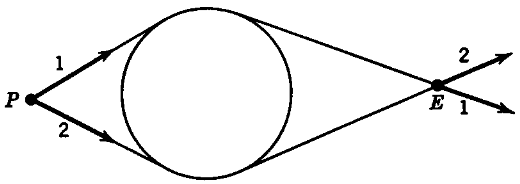


Figure 6.3.1

the elementary game with the matrix

$$\begin{matrix} 1 & -1 \\ -1 & 1. \end{matrix} \tag{6.3.1}$$

In the game of obstacle tag, for example, P and E may choose the optimal directions 1 or 2 of Figure 6.3.1. Here, say, E might choose 1, guessing that P will pick 1 and pursue by the upper route; E thus strives to match choice numbers. On the other hand, a mismatch is to P's advantage.

This dilemma of interdependent choices does not appear serious, for as soon as play progresses any positive amount one can expect that **x** will have moved off the dispersal surface. Thus we will resolve matters by what we term an *instantaneous mixed strategy* (IMS). Let some small positive ε be chosen. Let the players make their path decisions with the optimal strategies pertaining to the matrix (6.3.1), that is, choose their alternatives each with probability $\frac{1}{2}$. They persevere in these choices until the elapse of time ε. The position being no longer on the dispersal surface, the partie proceeds in the ordinary way, aside from a possible small penalty accruing from a wrong guess.

The indefiniteness of the "small" ε is undesirable, of course, in a theoretical problem of mathematical analysis. But in any practical application there will be a certain imprecision which will suggest a feasible value of ε: the amount of time, for example, required to detect and act upon the opponent's decision.

However, if K-strategies are adopted, there is no difficulty at all. The first decision (at the dispersal surface) is mixed, the rest are pure.

There can be dispersal surfaces which do not require an IMS. One player may be faced with a choice of optimal strategies when **x** is on the dispersal surface, but his opponent's may be unique. Then clearly the former player may take his choice indifferently.[3]

[3] An instance occurs in the bomber and battery game in the Appendix. When the bomber approaches a straight coast from a point on its normal from the battery O, he has two symmetrical, optimal routes. But as the battery's firing strategy is the same for either, the bomber gains nothing by mixing.

6.4. THE QUESTION OF THE PERPETUATED DILEMMA

The effect of an instantaneous mixed strategy, being of miniscule duration in practice, is not great. But there are cases where the need for mixing strategies appears to endure throughout an appreciable interim. We shall refer to such a phenomena as a *perpetuated dilemma*. The idea is perhaps best clarified by illustration and the following is typical.

Example 6.4.1. The wall pursuit game. Here P and E have simple motion with respective speeds w and 1 with $w > 1$.[4] The playing space $\mathscr{E}$ is a half-plane, its boundary W being the "wall" which E may not cross. We are interested here only in starting positions with E on W. Capture time is the payoff.

We will first take capture to mean the coincidence of P and E.

It can either be taken as obvious that E will do best by always remaining proximate to the wall W or we can adopt this limitation as a constraint. In the latter case, we have a problem of some intrinsic interest—the interception of an evader constrained (or assumed) to motion along a given straight line.

From a position as in Figure 6.4.1a, the optimal partie is clear: E will move upward and P will select the straight line path which will enable him to intercept E, say, at the point labeled C. It is well known and easy to see that with this "collision course" navigation by P, the straight line PE, as play progresses, retains the same slope. (Hence the soundness of "constant bearing navigation" against evaders constrained to a constant velocity.)

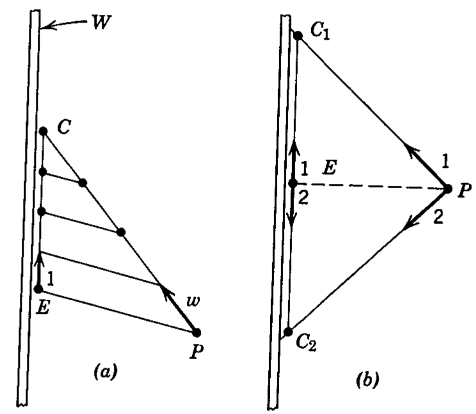


Figure 6.4.1

[4] Arbitrary speeds may be used with P's the greater; w can be construed as their ratio.

Our dilemma occurs when PE is perpendicular to W. Then both players are confronted with choices 1 and 2, ((b) of the figure) somewhat as held in obstacle tag. Supposing that play proceeds under congruous choices (such as 1 and 1), then the constant bearing principle shows that PE will remain perpendicular to W. Thus at each instant of play, the possibility is open to E of reversing his velocity without sacrificing optimality. The fact that P, always alerted for such a reversal, is similarly always faced with a corresponding pair of choices is our sought instance of the perpetuated dilemma.

The reader can possibly conceive of other simple games sharing this phenomenon. How should the players act? To mix strategies continuously throughout a positive interim seems theoretically absurd. But this difficulty is hardly grounds for jettisoning our science.

In many of the cases based on reality the trouble is merely caused by the problems being improperly set. If the terminal manifold is of smaller dimension than $n - 1$ (such is true of the example due to the stipulation of pointwise capture), the deficiency can be reflected as a perpetuated dilemma. Once the proper dimension is restored, the difficulty evaporates. The reason becomes apparent when the game is put into a suitable reduced space. We return to our illustration.

Let us change to a proximity capture, that is, restore a $\mathscr{C}$ of the proper dimension. We center P in the disk of radius l (Figure 6.4.2), the capture region. Again supposing perpendicularity at the outset, collision course navigation will carry E to some point C and P to P_c (which lies on the

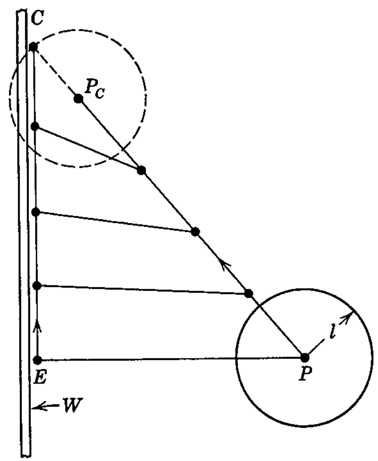


Figure 6.4.2

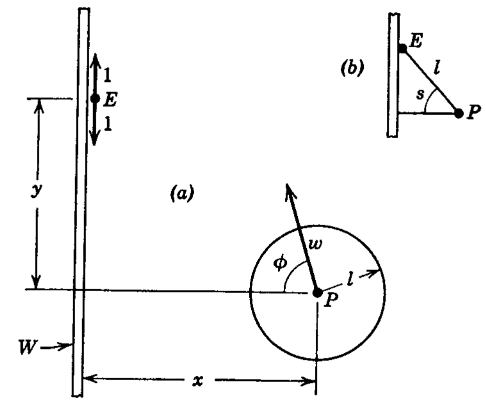


Figure 6.4.3

segment CP such that $|CP_c| = l$). Observe now, that progressively the segment PE changes its slope: it becomes less horizontal until at capture it becomes P_cC. Thus once the original perpendicularity is broken, it never recurs.

Despite the simplicity of this game, a formal analysis is enlightening. We shall require it in any case to illustrate the ideas of the next section.

It is clear that we can employ a reduced space with two coordinates x, y, as in Figure 6.4.3a.

The KE are obviously

$$\dot{x} = -w \cos \phi$$
$$\dot{y} = \psi - w \sin \phi$$

with
$$w > 1$$
and
$$-1 \leqslant \psi \leqslant 1$$
so that the ME_1 is

$$\min_{\phi} \max_{\psi} [-w(V_x \cos \phi + V_y \sin \phi) + V_y \psi] = -1.$$

Thus if $\rho = \sqrt{V_x^2 + V_y^2}$, then $\cos \bar{\phi} = V_x/\rho$, $\sin \bar{\phi} = V_y/\rho$, while $\bar{\psi} = \sigma = \operatorname{sgn} V_y$ and the ME_2 becomes

$$-w\rho + \sigma V_y = -1.$$

The RPE follow:

$$\mathring{x} = w\frac{V_x}{\rho}, \qquad \mathring{V}_x = 0$$

$$\mathring{y} = -\sigma + w\frac{V_y}{\rho}, \qquad \mathring{V}_y = 0.$$

Taking $s(-\pi/2 \leqslant s \leqslant \pi/2)$ as in (b) of the figure, $\mathscr{C}$ is described b

$$x = l \cos$$
$$y = l \sin s$$

so that
$$V_s = 0 = l(-V_x \sin s + V_y \cos s)$$

Hence
$$V_x = \lambda \cos s, \qquad V_y = \lambda \sin s.$$

As the Value gradient vector (V_x, V_y) is to point into the space, clearl $\lambda > 0$. We do not require its specific value. Finally,

$$\sigma = \operatorname{sgn} V_y = \operatorname{sgn} \sin s = \operatorname{sgn} s.$$

As $\mathring{V}_i = 0$, the V_i retain their values above and we are to integra

$$\mathring{x} = w \cos s$$
$$\mathring{y} = w \sin s - \sigma$$

which, with the initial conditions yield

$$x = (l + w\tau) \cos s$$
$$y = (l + w\tau) \sin s - \sigma\tau, \qquad \left(|s| \leqslant \frac{\pi}{2}, \tau > 0, \sigma = \operatorname{sgn} s\right) \qquad (6.4.$$

which describe the optimal paths. We shall return to these equations i the next section.

6.5. THE CONSTRUCTION OF DISPERSAL SURFACES

Let us suppose we are at the stage of the solution process where we hav just integrated the RPE from some set of initial conditions as might b borne, say, by $\mathscr{C}$. Suppose we find that the paths obtained fall into two classes which intersect (members of one class meet those of the other, no of their own) such as is suggested by Figure 6.5.1. We find the locus on which two paths, one from each class, meet and such that the Value at the meeting point is the same for both. This locus is the sought dispersal surface. Only the segments of the paths extending from the seat of the initial conditions to the dispersal surface are retained.

We illustrate with the preceding example. Returning to the equation (6.4.1), we see that the paths can be divided into two classes according as $s \gtrless 0$. We seek the points where two paths, one from each class, meet with the same Value which here is τ. That is for any fixed $\tau > 0$ we must have

$$x = (l + w\tau) \cos s_1 = (l + w\tau) \cos s_2$$
$$y = (l + w\tau) \sin s_1 - \tau = (l + w\tau) \sin s_2 + \tau$$

where
$$s_1 > 0 > s_2.$$

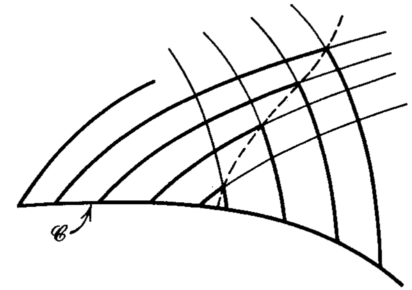


Figure 6.5.1

As $|s| \leqslant \dfrac{\pi}{2}$, the first equation implies $s_1 = -s_2 = s$ so that the second becomes

$$y = (l + w\tau) \sin s - \tau = -[(l + w\tau) \sin s - \tau]$$

implying that $y = 0$. Thus the x-axis (with $x > l$) is a dispersal surface and is the only such. The complete path map is sketched in Figure 6.5.2.

Recall that in our first version of this game, where capture meant the coincidence of P and E, we were beset by a perpetuated dilemma. To see how such arose, in Figure 6.5.2, let l, the radius of the capture region, become zero. From the figure it is clear that all paths such that

$$|\sin s| < w^{-1}$$

will then coincide with the dispersal surface and it becomes the renegade path.

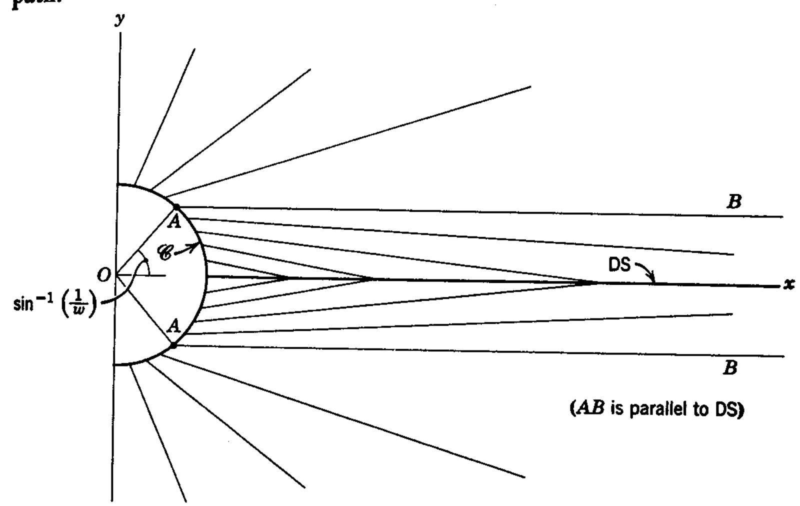


Figure 6.5.2

6.6. FURTHER EXAMPLES

In this and some later sections are further cases where the perpetuated dilemma is dispelled by use of an orthodox $\mathscr{C}$, but the means are dissimilar. Simple as is the example below, it does typify a situation which might well be embodied in more significant games.

Example 6.6.1. The one chance pursuit game (M. Dresher). Because this is a game of kind we will, to a slight degree, have to anticipate the techniques of Chapter 8.

Figure 6.6.1a shows the vectograms of P and E which are the same for all positions.[5] The desideratum of P is to attain capture (in the point coincidence sense); E to avoid it. It is clear that once P and E pass one another, the chance for capture is irrevocably lost.

It is when the line EP has a slope of 45° that the perpetuated dilemma occurs. Then P must continually outguess E in order to attain capture, and inversely for E to escape.

Taking x and y as the relative coordinates of P with respect to E, we can use them as reduced coordinates, as shown at (b). The vectograms for both players are identical. Capture consists of bringing $\mathbf{x}$ to the origin. The dotted line is one of perpetuated dilemma, for if E moves vertically, P must do so horizontally to stay on it and *vice versa*.

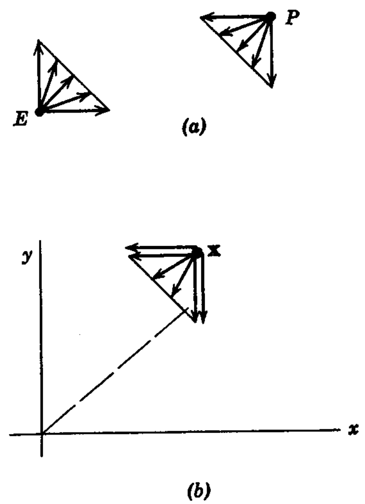


Figure 6.6.1

[5] We are taking the vectograms to be the same size, but only a slight modification is necessary if they are different but similar.

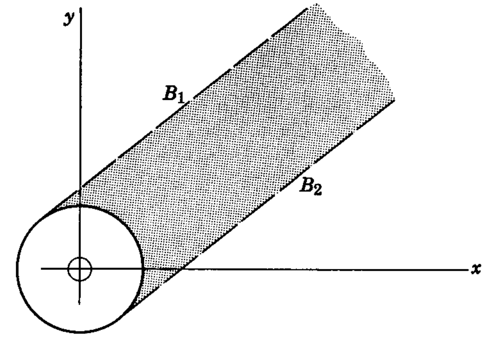


Figure 6.6.2

Let us see to what degree the contretemps is dispelled by employing a capture region of positive size. In Figure 6.6.2 it is taken as a disk.

From any point of the shaded region, between the lines B_1 and B_2, P can achieve capture. These lines are what we will later call *barriers*, and the reader will have a better understanding of their role in games of kind after having read Chapter 8. Here it suffices to note that if $\mathbf{x}$ is in the shaded region, P can keep it there by playing vertically when $\mathbf{x}$ is sufficiently close to B_1 and horizontally should $\mathbf{x}$ approach too close to B_2. This policy can easily be embodied into a definite strategy by assigning definite bounds to the closeness.

Entirely analagous remarks apply to E's behavior when $\mathbf{x}$ is outside the shaded region. He can prevent P from forcing the entry of $\mathbf{x}$ and so can achieve escape.

Thus the outcome of the game is precisely determined at all points save those on B_1 and B_2 themselves. But on these lines the strategy of each player is uniquely determined under penalty of definitely losing the game. On B_1, say, P must play his vertical velocity and E his horizontal, each having to act so as to keep $\mathbf{x}$ from shifting to the region unfavorable to him. Thus, once on B_1, the point $\mathbf{x}$ will traverse it. The result is that it will touch the capture disk without entering it.

Of course, we are at liberty to define such an outcome as either capture or escape. In Chapter 8 we shall give grounds for calling it neither, but terming it *neutral*.

But at all events, the perpetuated dilemma has been dispelled; unambiguous strategies can be defined for all $\mathscr{E}$.

Before proceeding with further examples, it is well to digress on

6.7. THE GEOMETRIC METHOD FOR SIMPLE PURSUIT GAMES OF KIND

The simple problem of the interceptor and bomber given in the first chapter (Example 1.9.2) is an illustration of the ideas to be somewhat expanded here. We deal with pursuit games and, for simplicity, ones with simple motion in the plane.

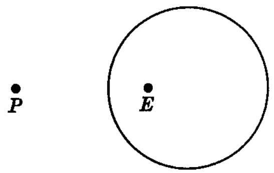


Figure 6.7.1

Let the set of points which E can reach without being captured, despite P's best efforts, be called the *safe region* and let the surface which bounds this region be designated by the BSR (boundary of safe region).

In many instances of pursuit games it is clear that under optimal play capture will occur on the point U of the BSR yielding the highest payoff to E.[6] The optimal strategies will be such that both players will travel to U in minimal time and capture will occur there.

These concepts have all been illustrated in Example 1.9.2 where the BSR is the perpendicular bisector of PE.

We wish to note here a few particulars about various types of BSR. If the ratio of E's speed to that of P is w, and capture is defined as coincidence of P and E, the BSR will be the set of points U, where

$$|EU| = w\,|PU|. \qquad (6.7.1)$$

For $w \neq 1$, this set is the well-known Appolonius circle (Figure 6.7.1). If $w < 1$, E is interior to the circle and P exterior. Remark the readily proved fact:

If P and E both travel straight toward a point U on the Appolonius circle, then any new such circle, obtained from a pair of simultaneous intermediate positions of P and E, is tangent to the original circle at U. (6.7.2)

If we replace coincidence by a circular capture region $\mathscr{C}$ of radius l, the BSR becomes the oval

$$|EU| = w[|PU| - l]. \qquad (6.7.3)$$

If $w = 1$, the locus of U becomes a branch of an hyperbola. Some geometric properties, easily proved, are worth noting.

[6] But not in all. If Example 1.9.2 is modified only by assigning a lower speed to P, then E can always reach the target.

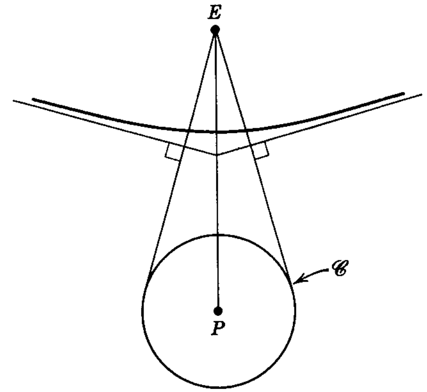


Figure 6.7.2

The hyperbola passes through the midpoint of E and the nearest point of $\mathscr{C}$ to E. The asymptotes pass through the midpoint of P and E and are perpendicular to the tangents from E to $\mathscr{C}$. Furthermore, P and E are the foci (*Figure 6.7.2*). (6.7.4)

Problem 6.7.1. Prove the assertion of the footnote: If P and E both have simple motion, but E is faster, P cannot forestall by prior capture E's reaching a given target. Capture means $|PE| \leqslant l$.

6.8. FURTHER EXAMPLES: THE FOOTBALL PLAYERS AND THE COOPERATIVE CUTTERS

Example 6.8.1. The football players. The vertical lines of Figure 6.8.1 are the sidelines of a football field. The ball carrier E desires to move as far upward—toward his goal—as possible. He is opposed by the single tackler P. Both players have simple motion with the same speed. Capture is positional coincidence.

The problem can be handled in the manner of Example 1.9.2: We draw the perpendicular bisector, the BSR, of PE and seek its uppermost point U in bounds. Both players run toward U, which generally lies on a sideline.

Our dilemma occurs when the segment PE is vertical. The bisector now parallels the goal lines, and all its points are equally meritorious as destinations. There are not merely two but a continuum of equally good choices confronting both players.

One is tempted to offer the solution: P should always remain at the mirror image E about the bisector. But this policy is not a strategy, for P

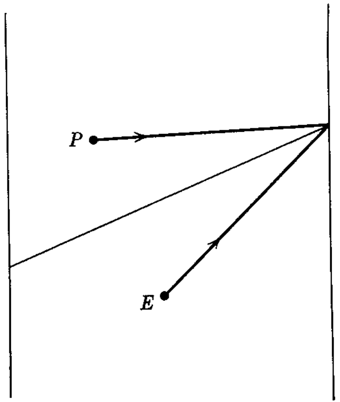


Figure 6.8.1

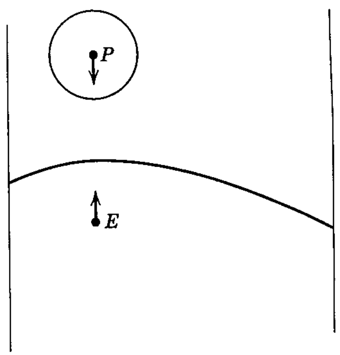


Figure 6.8.2

would have to base his directional decision of E's velocity, a control variable.[7] We have discussed earlier our motives for rejecting such procedures.

As before, we alter capture to the entry of E into a disk of radius l centered at P. As stated in the last section, the bisector is replaced by the arc and hyperbola. When PE is vertical, the quandary evaporates; E and P should each run directly forward (Figure 6.8.2).

Note the degree of resolution here. We have passed from a situation of infinitely many choices to one with no ambiguity at all! Even an instantaneous mixed strategy is not required.

If E finds himself underneath P, within the vertical projection of the capture circle, the geometric analysis would indicate that the sidelines would generally play no part in the solution. For the result (6.7.4) shows that in this case, the two asymptotes would slope downward as they progress from the midpoint of PE out toward the sideline. If the sidelines are at an appreciable distance from P and E, the hyperbola branch will also slope thus, and its highpoint must be well within the ball field.

The formal analysis below will confirm this statement. It leads to a very simple solution free of any reference to algebraic curves such as are the BSR. We dispense with the sidelines and, on the grounds of this slight increase in generality, assign a new title.

[7] However legitimate strategies exist which give arbitrarily close approximations. If P starts from a position near but not at the mirror image, he can adopt the strategy of always traveling toward it.

Example 6.8.2. The simple blocking game. The state and control variables are as in Figure 6.8.3 so that the KE are (the speeds being unity)

$$\dot y_1 = -\cos\phi$$
$$\dot y_2 = \cos\psi$$
$$\dot x = -\sin\phi - \sin\psi.$$

We have a terminal payoff with $H = y_2$.

The ME_1 is

$$\min_{\phi}\max_{\psi}\,[-(V_1\cos\phi + V_3\sin\phi) + (V_2\cos\psi - V_3\sin\psi)] = 0$$

so that if $\rho_1 = \sqrt{V_1^2 + V_3^2}$, $\rho_2 = \sqrt{V_2^2 + V_3^2}$

$$\cos\bar\phi = \frac{V_1}{\rho_1}, \quad \sin\bar\phi = \frac{V_3}{\rho_1}; \qquad \cos\bar\psi = \frac{V_2}{\rho_2}, \quad \sin\bar\psi = \frac{-V_3}{\rho_2}.$$

Thus the ME_2 is

$$-\rho_1 + \rho_2 = 0.$$

We have for $\mathscr{C}$

$$y_2 = s_2, \quad x = l\sin s_1, \quad y_1 = s_2 + l\cos s_1$$

(s_1 is angle of PE and the vertical.) As on $\mathscr{C}$, $V = s_2$

$$V_{s_1} = H_{s_1} = 0 = l(V_3\cos s_1 - V_1\sin s_1)$$
$$V_{s_2} = 1 = V_1 + V_2.$$

Thus

$$V_1 = \lambda\cos s_1, \quad V_3 = \lambda\sin s_1, \quad V_2 = 1 - \lambda\cos s_1.$$

If $s_1 = 0$, so that capture has occurred with E at the lowermost point of $\mathscr{C}$, then $V_1 = \partial V/\partial y_1 > 0$ as we can see that increasing y_1 (with y_2 and x fixed) increases V. Thus $\lambda > 0$. The ME_2 gives

$$\rho_1 = \lambda = \rho_2 = \sqrt{(1 - \lambda\cos s_1)^2 + \lambda^2\sin s_1^2}$$

from which

$$\lambda = \frac{1}{2\cos s_1}.$$

At this point we can note the optimal strategies and adduce a complete picture of a partie without integrating.

$$\cos\bar\phi = \frac{V_1}{\rho_1} = \frac{\lambda\cos s_1}{\lambda} = \cos s_1$$
$$\sin\bar\phi = \sin s_1$$
$$\cos\bar\psi = \frac{1 - \lambda\cos s_1}{\lambda} = \cos s_1$$
$$\sin\bar\psi = -\sin s_1$$

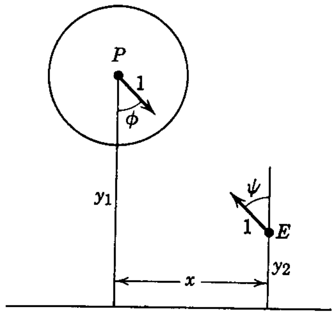


Figure 6.8.3

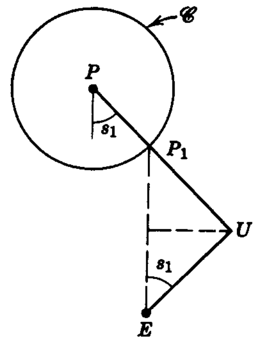


Figure 6.8.4

so that

$$\bar{\phi} = s_1 = -\bar{\psi}$$

and the optimal paths must appear as in Figure 6.8.4. As the equal speeds imply $|P_1U| = |EU|$, and as the two angles s_1 are equal, it follows that EP_1 is vertical. The simple construction for the optimal strategies follows:

From E draw a vertical line meeting $\mathscr{C}$ at P_1 (the lowermost intersection). Draw PP_1 and extend it to U, where it meets the perpendicular bisector of EP_1. Both players head toward U.

Should E lie without the vertical projection of $\mathscr{C}$, the second remark of (6.7.4) about the slope of the hyperbola's asymptotes shows that by steering for a sufficiently remote point on this curve he can attain as large a payoff as he pleases. Thus no solution then exists.

It is easy to see that this solution jibes with the geometric one. First U lies on the hyperbola from the definition of the latter as a locus. That the tangent is horizontal at U follows from the well-known property of a hyperbolic mirror's reflecting a light ray emanating from one focus as if it came from the other focus.

Exercise 6.8.1. Find the Value of the football problem when it is a composite of Examples 6.8.1 and 6.8.2. That is, the field has sidelines *and* there is a positive capture radius. As before, both P and E have simple motion with the same speed. The sought Value is the gain (of distance toward his goal) that E can make with both players acting optimally.

Example 6.8.3. The two cutters and the fugitive ship. There are two pursuers (the cutters) P_1 and P_2 chasing a fugitive E. All move with simple motion, the speeds of the cutters each being greater than that of E.

Time of capture is the payoff and coincidence of either P_1 or P_2 with E capture.

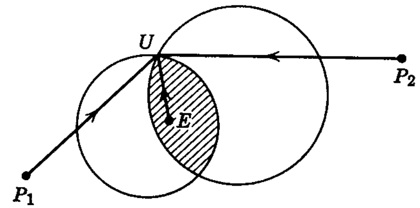


Figure 6.8.5

The geometric solution proceeds as follows: Draw the two Appolonius circles (re P_1 and E, P_2 and E). The intersection of their disks (shaded in Figure 6.8.5) being the region in which E is safe from both pursuers, he heads for its most distant point U and so do P_1 and P_2.

Cieba[8] has given a rigorous proof, along lines quite different from ours, that this solution is correct.

The perpetuated dilemma occurs when P_1EP_2 lie on a straight line in that order. Then there are two points U both equidistant from E. Toward which should all three craft head? If they all pick the same one, they continue during play to remain collinear and, by the remark (6.7.2), the points U are unchanged. We have optimal play but with a perpetuated dilemma.

Cieba's resolution is to have the pursuers take E's velocity as well as position into account and is analogous to the mirror image policy of the tackler in last example and open to the same objections.

But again we equip each pursuer with a positive capture radius. It is now clear that once the chase is under way, the three points lose their collinearity and the dilemma reduces to an IMS.

Cieba points out that the symmetric case, with equally fast pursuers, with E midway between them, is equivalent to the wall pursuit game (Example 6.4.1). For if in this game we place a second pursuer, as a mirror image to the first, about the wall, the latter constraint may be removed, symmetry compelling E to follow his old course. Thus in this case of the two cutter game, at least, we have already done a full analysis.

6.9. THE EXISTENCE OF THE PERPETUATED DILEMMA

Can such existence occur in a legitimately formulated game? The answer is affirmative, as shown by

Example 6.9.1. A game with perpetuated dilemma. For $\mathscr{E}$ we take the half-plane above the x-axis which shall be $\mathscr{C}$. The payoff is to be terminal

[8] We have read the ms. of his paper on pursuit games but do not know if and where it was published.

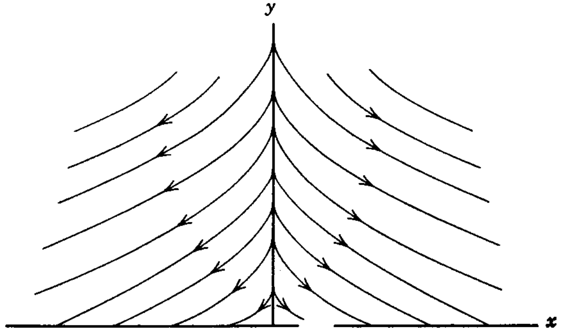


Figure 6.9.1

with $H(x)$ differentiable and even; it has a maximum when $x = 0$ and decreases as $x \to \pm\infty$. The KE are

$$\dot{x} = \phi(1 + 2\sqrt{|x|}) + \psi, \qquad -1 \leqq \phi, \psi \leqq 1$$
$$\dot{y} = -1.$$

Thus **x** has a fixed downward velocity and must meet $\mathscr{C}$; E desires it to do so as near $x = 0$ as he can; P, as far. The ME_2 is

$$\bar{\phi}(1 + 2\sqrt{|x|})V_x + \bar{\psi}V_x - V_y = 0$$

where $\bar{\phi} = -\text{sgn } V_x$, $\bar{\psi} = \text{sgn } V_x$ and the RPE are simply

$$\mathring{x} = -\bar{\phi}(1 + 2\sqrt{|x|}) - \bar{\psi}, \qquad \mathring{V}_x = -V_x/\sqrt{|x|}$$
$$\mathring{y} = 1 \qquad \mathring{V}_y = 0.$$

At $\mathscr{C}$

$$x = s, \quad y = 0, \quad V = H(s)$$

and symmetry enables us to work with $s > 0$ only.

Then on $\mathscr{C}$

$$V_x = V_s = H'(s) < 0$$

and so $\bar{\phi} = 1$, $\bar{\psi} = -1$. From the relevant RPE it is clear that V_x will not change sign and so these strategies persist. Integrating the system

$$\mathring{x} = -2\sqrt{x}, \qquad \mathring{y} = 1$$

yields the paths

$$y = \sqrt{s} - \sqrt{x}.$$

We complete the picture symmetrically in Figure 6.9.1. Clearly the upper y-axis is a dispersal surface. On it both players have the choice of any horizontal velocity of modulus $\leqslant 1$. The maximizing player E will want **x** to remain on the DS and so will try to choose a velocity opposite to P's selection. On the other hand, P strives for concordant choices. Thus as long as E succeeds in keeping **x** on the DS, there is a perpetuated dilemma.

Even here we do not feel a serious conceptual hindrance. For suppose we quantize the game so that the players act in a discrete sequence of small, closely spaced moves, each a miniscule game with mixed optimal strategies; E loses once **x** moves off the DS, never to return. And the probability of each such loss is $\frac{1}{2}$. Therefore it is extremely improbable that **x** stays on the DS for very long. If the continuous case is regarded as the limit of finer quantizations, it appears as valid to accept immediate removal. Certainly we may do so in practical problems.

Problem 6.9.1. Show if the only change in the last example is to replace the first KE by

$$\dot{x} = \phi(1 + x^2) + \psi$$

the paths are as in Figure 6.9.2. Thus it is possible to have a path of perpetuated dilemma that is not a dispersal surface.

In Figure 6.9.2 there are no solutions outside the curves A. From such starting points P can obtain an arbitrarily small payoff. Thus the A are singular surfaces of type $(0, u, p)$.

It is easy to see that the paths near any path K of perpetuated dilemma must have the same direction as K. That is, if K is a DS the branch paths must meet in tangentially. This fact could have been used in our earlier examples to preclude the perpetuated dilemma directly, for there the paths were straight lines.

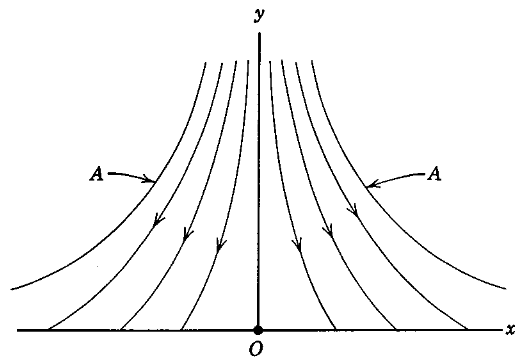


Figure 6.9.2

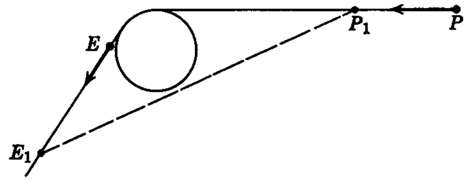


Figure 6.10.1

6.10. VARIOUS PROBLEMS

We mentioned earlier unsuspected intransigencies of differential games. As a case in point, we return to obstacle tag. This pursuit game was coined merely to introduce the subject of dispersal surfaces with as simple and obvious an example as possible. For such starting positions as sketched in our earlier figures, the solution is manifest enough not to require analysis. Yet for others the game is not so innocuous.

Suppose E starts from very near the obstacle and P reasonably far away, as shown by E and P in Figure 6.10.1. If the players now act according to our former "obvious" optimal strategies, they will soon be at positions such as E_1, P_1. Here the obstacle no longer intervenes! The "obvious" solution is obviously incorrect, for the broken line is clearly a better path. None of the ideas in this book appears adequate to solve this game.

Research Problem 6.10.1. What is the full solution to the game of obstacle tag?

The problem below entails a dispersal surface determined from means other than mere symmetry.

Problem 6.10.1. In the game with KE:

$$\dot{x} = (\phi - a\psi)u(x)$$
$$\dot{y} = -u(x), \qquad -1 \leqslant \phi, \psi \leqslant 1$$

where $0 \leqslant a < 1$ and $u(x)$ is positive, has a minimum at $x = 0$, is otherwise monotonic, is infinite at $x = \pm\infty$ and $\int_{-\infty}^{\infty} 1/u(x)\, dx$ exists.

Here $\mathscr{E}$ is $y \geqslant 0$ and $\mathscr{C}$ is $x = s$ $(\infty < s < \infty)$, $y = 0$. The payoff is integral with $G = 1$.

Prove: 1. There exists exactly one dispersal surface.
2. It meets the x-axis at 0, lies within the sector

$$|x| \leqslant (1 - a)y$$

and is asymptotic to the line $x = k$, where k satisfies

$$\int_{-\infty}^{k} \frac{dx}{u(x)} = \int_{k}^{\infty} \frac{dx}{u(x)}.$$

3. In the symmetric case where $u(x) = u(-x)$ the above surface will be $x = 0$, $y > 0$, and on it V_x is not zero and discontinuous.

The following problem is difficult. As a military problem its ultility appears slight because of the miniscule advantages of the payoff, but it has some novel features, the understanding of which would advance the present theory.

On the other hand, a game of kind version of this situation would be of immense practical importance if it is reasonably possible for the aircraft to dodge the missile by veering as described below. There would, of course, have to be some limitation on the missile's turning capability, but he should navigate as best he can to anticipate the veering of his target.

Research Problem 6.10.2. There are certain types of antiaircraft missiles which, due to physics of the means employed, can sense the proximity of their target much better from its rear than from its front. For example, in Figure 6.10.2a we will take the oval drawn about the airplane E as delimiting the region in which the missile P can detect his quarry. In terms of our theory, then, the oval will correspond to the capture surface and we will denote its boundary by $\mathscr{C}$.

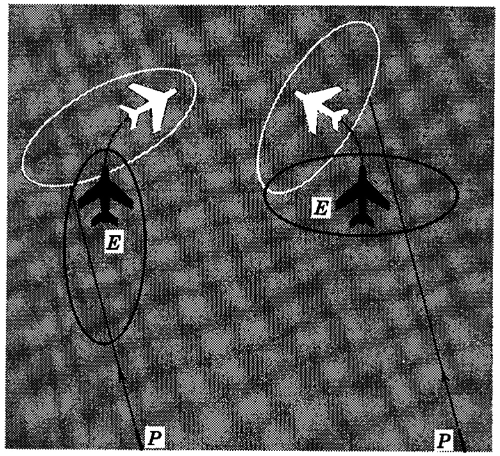
Let the airplane desire to advance as far as possible before capture, that is, P's entry in $\mathscr{C}$. In the position sketched, it seems plausible that E can achieve a gain by veering to the right when capture is imminent and so swinging $\mathscr{C}$ away from P. Should $\mathscr{C}$ be shaped as at (b), it appears that a similar gain would be witnessed by E's veering left.

Besides the usual aspects of a problem in differential games, this situation generates some unusual questions.

Which way should the aircraft veer? [We have answered this: The veering is left or right (with P on the right as in the figure) depending on whether $dD/ds >$ or < 0 at the point where P will meet $\mathscr{C}$. Here D is the distance from E to a point of $\mathscr{C}$ and s the arc length of $\mathscr{C}$ measured from the centerline. [Compare (a) and (b) of the figure).]

It seems plausible that when P is far away, E flies straight. Just when does he commence to turn?

The dispersal surface would be among the set of symmetric positions with P directly behind E's tail. Is an IMS involved in one part of the DS (P and E close) and not on another (P and E remote)?



(a) (b)

Figure 6.10.2

Example 6.10.1. Sandpiles. If as much sand as possible is heaped upon a flat horizontal plate of arbitrary shape, the upper surface of the sandpile will have almost everywhere the same gradient, which is a characteristic constant of the sand. For example, a circular plate supports a cone and an oblong one, a hip roof. The crests formed by the five roof peaks are fairly obvious, but more irregular plates can yield some surprising configurations.

Formally, if $u(x, y)$ is the height of the pile, it must satisfy,[8] except at crests,

$$u_x^2 + u_y^2 = 1 \tag{6.10.1}$$

and $u = 0$ at the plate boundary, a closed curve $\mathscr{C}$.

Now consider the one-player, differential game: the point $\mathbf{x}$, initially within $\mathscr{C}$, moves with simple motion and unit speed and it is to reach $\mathscr{C}$ in minimal time. We assert:

The plot of V for the game is the upper surface of the sandpile and the crests of the latter (in planform) are the dispersal curves of the game.

For once again the *KE* are

$$\dot{x} = \cos\phi, \quad \dot{y} = \sin\phi$$

and a simple analysis yields

$$\cos\bar{\phi} = -V_x/\rho, \quad \sin\bar{\phi} = -V_y/\rho, \quad \rho = \sqrt{V_x^2 + V_y^2} \tag{6.10.2}$$

[8] We have normalized the above sand constant to 1.

and the ME_2 is easily seen to be

$$-\rho + 1 = 0$$

so that V satisfies (6.10.1). Also $V = 0$ on $\mathscr{C}$.

To construct the solution, note that the RPE contain $\mathring{V}_x = \mathring{V}_y = 0$, which, with (6.10.2) imply that the optimal paths are straight. Denoting a smooth portion of $\mathscr{C}$ by $\mathbf{x} = X(s)$, $y = Y(s)$, our initial condition equation (on $\mathscr{C}$)

$$V_s = 0 = X'V_x + Y'V_y$$

implies that the paths meet $\mathscr{C}$ at right angles. As distance and time ($= V$) agree on the paths (as speed $= 1$), we apply the construction of Section 6.5 to obtain the DS, utilizing simply distance along the interior normals to $\mathscr{C}$ as V. As V is the height of the plot, the latter is a continuous surface which must agree with the top of the sandpile.

Problem 6.10.2. Study the geometry of these dispersal curves. For example, with polygonal $\mathscr{C}$, they are comprised of pieces of straight lines and parabolas. If a smooth $\mathscr{C}$ has a "vertex" (a point of locally minimal radius of curvature) the DS terminates at the center of the osculating circle there.

CHAPTER 7

Afferent or Universal Surfaces

7.1. INTRODUCTION

As stated in the last chapter the singular surfaces of the title are of type $(+, u, +)$, and the typical one in three dimensions has been depicted in Figure 6.1.1b.

To envisage the role of universal surfaces, abbreviated US, in differential games, one can think of such a surface as a union of especially advantageous paths. Optimal play will demand that the descriptive point $\mathbf{x}$ be brought to the US and thereafter remain on it.

We are dealing with games, in which the essence is that the interests of the players are conflicting. Therefore the word "advantageous" above, if applicable to one player, can be expected to be the contrary to his opponent. Thus, often the decision to utilize a certain surface as a US rests, under optimal play, with one player alone. It even may be that his opponent's optimal strategy is perfectly continuous on the universal surface and nearby.

Consequently, much of our investigation will treat of one-player games. Little generality is lost thereby, for we can think of the opponent as operating under the aegis of his optimal strategy that has already been set. Later we shall ascertain this setting and so restore the competitive aspect.

Accordingly we single out for definition: A *ϕ-universal surface* (or ϕ-US) is one which is created by a discontinuity in $\bar{\phi}$, while $\bar{\psi}$ is continuous on the surface. An *ψ-universal surface* (ψ-US) is defined similarly (ϕ and ψ interchanged).

What appears to be the most interesting type of US occurs when the

kinematic equations are linear at least in some of the control variables (and G also in case the payoff is integral). Let ϕ_k be such a variable.

Then the idea of limiting ourselves to one-player games can be pushed further; we may deal with games with only one control variable. The idea is, as before, that we assume all other than ϕ_k replaced by the $\bar{\phi}_j(x)$, $\bar{\psi}_j(x)$ which are constituents of optimal strategies. In the resulting one control variable game, the main equation will be linear in ϕ_k as all the KE (and G) are. Let A be the coefficient of ϕ_k therein. Then generally, under optimal play, ϕ_k will assume one or the other of its extreme values allowed by the constraints according as the sign of A. The only possibility of ϕ_k's assuming an intermediate value occurs at points where $A = 0$.

Let us suppose there to be a surface $\mathscr{S}$ such that $A = 0$ on $\mathscr{S}$ but nowhere else in some neighborhood of $\mathscr{S}$. Such loci alone can be universal surfaces. However, they need not be. If A does not change sign across $\mathscr{S}$, we should not expect $\mathscr{S}$ to be singular at all. If it does, $\mathscr{S}$ may be a transition or dispersion surface. The former we have seen actually to occur in some of our examples; on the latter, on the other hand, the V_j themselves may be discontinuous and A need not exist at all.[1]

Unlike most other types of singular surfaces which we treat in this book, the US involve no *retrograde* paths leading *to* them. Therefore their detection cannot follow from the integration of the RPE, and their possible locations are largely independent of $\mathscr{C}$ and the initial conditions.

We do not give a full theory. Indeed the subject of linear vectogram US appears big, possibly as big, as our later discussion will indicate, as the major chapters of the calculus of variations. We will give an analytic necessary condition for the cases up to four dimensions. It suffices for many problems, but there appear to be many other interesting facets which we leave unexplored.

Not all universal surfaces require linear vectograms. We take up another type first in the section immediately following because of its historical contiguity with the calculus of variations. They appear to be of secondary interest, and the reader may skip the next section with no loss of future comprehension.

7.2. UNIVERSAL SURFACES WITH NULL INTEGRAND

Supposing, in a game with integral payoff, the integrand G depends on the x_i only, is 0 on a certain surface $\mathscr{S}$ but is otherwise positive. Then $\mathscr{S}$ constitutes a locus of free rides for the minimizing player P, in that, once

[1] See Examples 7.3.1 and 7.5.1, where $A = uV_x \neq 0$ near the dispersal surface.

on $\mathscr{S}$, travel of $\mathbf{x}$ incurs no penalty for him as an increased payoff. Thus, clearly, there will exist cases where $\mathscr{S}$ will be a ϕ-US.

The analyses of one-player (P) games in such cases is quite straightforward.

Clearly V will be constant on $\mathscr{S}$. If $\mathscr{S}$ meets $\mathscr{C}$, this constant will be zero. If not, we apply our usual methods, calculating $V(\mathbf{x})$, beginning at $\mathscr{C}$ and working away from it until $\mathscr{S}$ is encountered. The lowest value of V thus found on $\mathscr{S}$ will be the constant. From the point (*s*) U where this minimum occurred, the optimal path(s) will leave $\mathscr{S}$.

Then, using $\mathscr{S}$ as a seat of initial conditions, we find, by the usual integration of the RPE, the paths leading into $\mathscr{E}$.

Under optimal play, $\mathbf{x}$ will traverse one of these paths to $\mathscr{S}$, then, by any admissable route whatever, proceed to U, and thence utilize an already found optimal path to $\mathscr{C}$.

The only possible novelty here lies in our point of view; the substance is very old. We illustrate by treating a famous stock problem of the calculus of variations from the universal surface standpoint.

Example 7.2.1. Surface of revolution of least area. Given two points in the plane both above the x-axis, to find the curve joining them which, when rotated about the x-axis, generates the surface of minimal area.

The solution is classic fare. When the points are sufficiently near one another and far from OX, the curve is a catenary. Under sufficient negation of these conditions, the curve becomes the union of three segments: two extending vertically from the points to OX and the segment of OX joining their feet: the well-known "Goldschmidt discontinuous solution."

We relook at matters.

In our one-player game, $\mathbf{x}$ moves with simple motion at unit speed. Thus we take as KE

$$\dot{x} = \cos\phi$$

$$\dot{y} = \sin\phi.$$

Since the elementary formula for area of revolution generated by an arbitrary curve is $2\pi\int|y|\,ds$, we take

$$G = |y|$$

so that our payoff will be proportional to the desired area. We use $|y|$, not y, so that we can use as $\mathscr{E}$ the entire x, y-plane.

The ME_2 is

$$-\rho + |y| = 0$$

and the RPE are

$$\mathring{x} = \frac{V_x}{\rho}, \qquad \mathring{V}_x = 0$$

$$\mathring{y} = \frac{V_y}{\rho}, \qquad \mathring{V}_y = \operatorname{sgn} y$$

where

$$\rho = \sqrt{V_x^2 + V_y^2},$$

and also

$$\cos\bar{\phi} = -\frac{V_x}{\rho}, \qquad \sin\bar{\phi} = -\frac{V_y}{\rho}.$$

The choice of end conditions is at our disposal. The familiar version with fixed endpoints we relegate to an exercise. Our curves will extend from some point on the y-axis to a given right endpoint. Thus, in accordance with the present theory, we take

$$\mathscr{E}:\ x \geqslant 0, \quad -\infty < y < \infty$$

$$\mathscr{C}:\ x = 0, \quad y = s.$$

First, our standard construction of optimal paths emanating from $\mathscr{C}$ will yield the classical catenaries. We shall find them in the upper half-plane only and so take $y \geqslant 0$, $s \geqslant 0$.

As on $\mathscr{C}$, $V = 0$, we have $V_s = 0 = V_y$, and we complete the initial conditions, using the ME_2 (clearly $V_x \geqslant 0$ from the nature of the problem):

$$V_x = |s| = s, \qquad V_y = 0.$$

The integration of the RPE is straightforward, yielding

$$V_x = s, \qquad V_y = \tau$$

and

$$x = s\log\frac{(\rho + \tau)}{s}, \quad y = \rho, \quad \rho = \sqrt{s^2 + \tau^2} \qquad (7.2.1)$$

the sought paths. Elimination of τ leads to the equation of a catenary, $y = s\cosh(x/s)$. Of course, the original phrasing of the problem requires that s be selected so that the path (7.2.1) passes through the prescribed starting point.

We will need below

$$V = \int_0^\tau \rho\,dt = \frac{1}{2}\left[\rho\tau + s^2\log\frac{\rho + \tau}{|s|}\right]. \qquad (7.2.2)$$

Now for our major interest, the ϕ-US. The curve $\mathscr{S}$ on which $G = 0$ is

$$x = \mu, \qquad y = 0$$

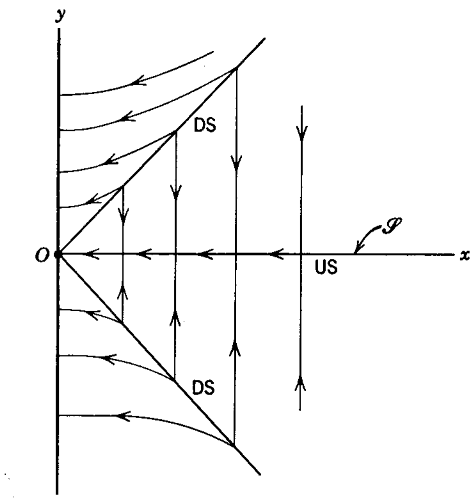


Figure 7.2.1

where μ, a parameter, $\geqslant 0$. The ME_2 shows that on $\mathscr{S}$

$$V_x = V_y = 0.$$

(As $\mathscr{S}$ meets $\mathscr{C}$, on the former, $V = 0$.) Integrating the RPE with these initial conditions gives us, for paths emanating from $\mathscr{S}$:

$$V_x = 0, \quad V_y = \sigma\tau, \quad \rho = \tau$$
$$x = \mu, \quad y = \sigma\tau.$$

Here $\sigma = \operatorname{sgn} y$ and clearly we use both $\sigma = 1$ and -1. These "tributaries" to $\mathscr{S}$ are clearly vertical lines. We compute directly that on them

$$V = \tfrac{1}{2}\tau^2 = \tfrac{1}{2}y^2. \tag{7.2.3}$$

The two sets of paths join on a dispersal surface which is obtained by equating the two values of V, (7.2.2) and (7.2.3); in the upper half-plane

$$\tfrac{1}{2}y^2 = \tfrac{1}{2}\left[\rho\tau + s^2 \log \frac{\rho + \tau}{s}\right].$$

If from this equation and the pair (7.2.1) we eliminate s and τ, we reach the equation of the upper dispersal surface, which turns out to be

$$x = cy.$$

Here c is a constant satisfying

$$c \exp \tfrac{1}{2}(1 + c^2) = 1, \qquad (c = .53\ldots).$$

The optimal paths are sketched in Figure 7.2.1.

Exercise 7.2.1. The classical problem with two fixed endpoints. Let $Q = (a, b)$ and we use for $\mathscr{C}$

$$x = a + \delta \cos s$$
$$y = b + \delta \sin s$$

so that

$$\delta(-V_x \sin s + V_y \cos s) = 0.$$

Proceeding in our customary manner and taking the limit as $\delta \to 0$, we have on $\mathscr{C}$

$$V_x = b \cos s$$
$$V_y = b \sin s.$$

Complete the solution using these initial conditions.

Research Problem 7.2.1. The two-handed version. Let Q above move with simple motion, but with speed <1, under the control of E, so that the problem becomes a pursuit game. Under certain circumstances, clearly E can control which type of path (catenary or 3-piece polygonal line) P will follow. Does the optimal solution reflect this control's being a major factor in E's strategy?

7.3. UNIVERSAL SURFACES WITH LINEAR VECTOGRAMS: AN INTUITIVE PURVIEW

To illumine our general ideas we consider one-player, two-dimensional games. Such must necessarily have integral payoffs if there is to be a universal surface. For, if the payoff were terminal, we know that the Value is constant on each optimal path. Then it is constant on the US and must retain the same value on all tributary paths; thus it is constant in a region containing the US. Accordingly in this region all strategies are optimal, and the US exists only in the most trivial sense.

Having settled on an integral payoff, we shall assume, in virtue of the last section, that $G \neq 0$. Let us take for definiteness G always positive. It will simplify matters if G is independent of the single control variable, ϕ, but we shall soon see that the following ideas hold when G is linear in ϕ.

Suppose in a region of $\mathscr{E}$, the gradient of V exists and is not zero. (Note that this is not true near a US of the null G type.) Then we will show, in a rough geometrical way, that a US can arise only if the vectograms are linear.

Let Figure 7.3.1a depict a typical linear vectogram. Imagine it drawn to a very small scale so that its vectors closely approximate the actual possible displacements of $\mathbf{x}$ during a very short interim. Superposed on the

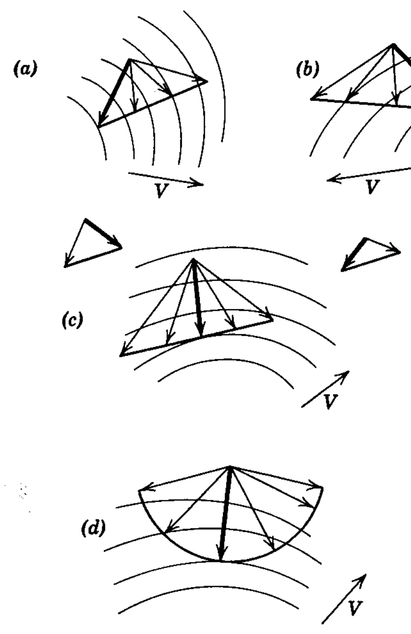


Figure 7.3.1

same figure are some of the curves on which V is constant; V is supposed to increase in direction of the arrow labeled V.

Which vector of this vectogram is best? Obviously the one achieving the greatest decrease in V or the one penetrating farthest through the pencil of curves. In this case it is the (over-scored) leftmost vector.

In (b) we have a similar situation, but now the greatest decrease of V accrues to the rightmost vector.

It is clear that the criterion is the direction in which V increases along the headline (the line of the arrowheads). Thus an intermediate vector can attain maximal penetration only when this increase is zero or when the headline is tangent to the local curve of constant V. Such a case is depicted at (*c*).

Suppose there is a curve on which such occurs at each point. For example, the dashed curves of Figure 7.3.2, where it is supposed that all vectograms have horizontal headlines. Such curves alone are eligible to be universal. Let us now return to Figure 7.3.1c and suppose that the point depicted lies on such a curve. From our study of (a) and (b) of this figure, we see how the vectograms immediately to the right and left will behave: the small vectograms drawn at (c) have their optimal vectors overscored.

Thus the optimal paths from both sides will converge to the center and we will have a US.[2]

To justify our assertion that linear vectograms are essential, observe a typical properly convex vectogram such as appears in Figure 7.3.1d. Again the criterion of matching the slopes of headline and V-curve holds; in the figure the overscored vector yields maximal penetration. Thus, generally, the minimizing ϕ is an interior one and varies continuously with $\mathbf{x}$ so that no US can appear. If the vectogram is but slightly convex, we may expect that the optimal paths will resemble those of the linear case, probably in some manner such as that suggested by Figure 7.3.3.

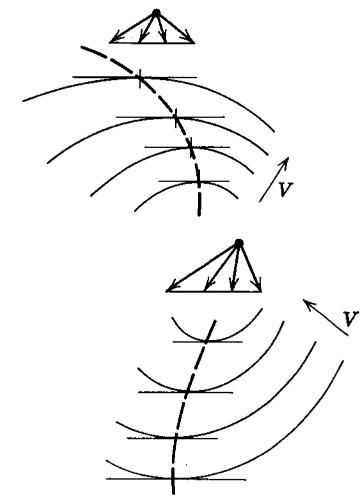


Figure 7.3.2

George Dantzig has suggested exploiting this idea to compute a US. Render a linear vectogram slightly convex, ascertain the paths, and then study their limiting behavior as we allow the convexity to disappear.

Let us seek some heuristic insight as to what may happen when more control variables are present or at least considered active. Figure 7.3.4 depicts a three-dimensional analogue of the subject of Figure 7.3.1. The

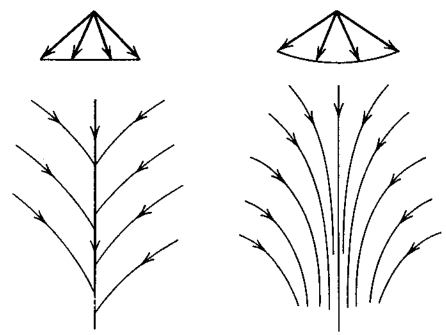


Figure 7.3.3

[2] Should the V-curves curl upwards, however, the little vectograms would have to be interchanged and we would have a dispersal surface. But generally at such the V_i are discontinuous and the V curves suffer a saltus in their slopes. We invite the reader to attempt the construction of an example in which there are smooth V-curves at a dispersal surface.

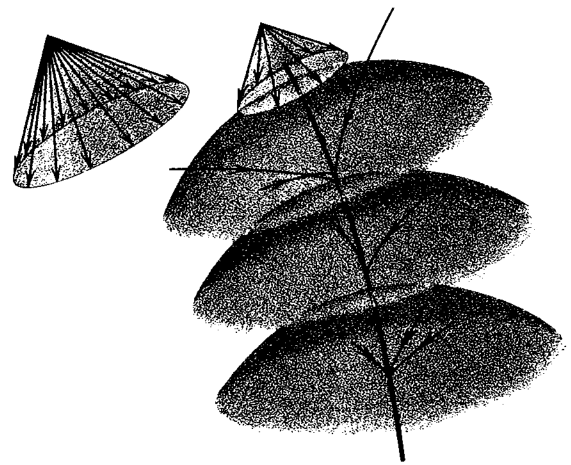


Figure 7.3.4

vectograms are cones and linear in the sense that their bases are flat (the headplanes). Also drawn are some surfaces of constant V. The same argument as before shows that an afferent behavior may possibly occur on the curve on which the headplane of the local vectogram is tangent to the V-surface. Then we have a "*universal curve*" rather than a surface.

It seems plausible that this is an instance of a general truth: vectograms linear in several control variables can lead to universal manifolds of lower dimension. It is interesting to think of the possibility of such a manifold's consisting of the intersection of a number of US each relating to a distinct control variables. We have not explored the ample terrain which appears to spread before us here.

An example may elucidate our feel for the subject a bit further.

Example 7.3.1. Let us consider motion in the upper half-plane with P endeavoring to have $\mathbf{x}$ reach the x-axis in minimal time. All the vectograms are to be shaped as in Figure 7.3.5a, but their size is a given smooth function u of x and y. That is, the KE will be

$$\dot{x} = \phi u(x, y)$$
$$\dot{y} = -u(x, y), \qquad -1 \leqslant \phi \leqslant 1, \quad u > 0.$$

Let us think of the surface which is the graph of u as being roughly indicated in the figure, where the curves are meant to be sections perpendicular to the page. That is, U_1 and U_2 are crests of ridges while D lies at the bottom of a valley.

Clearly the optimal paths, when nonsingular, will have slopes of $\pm 45°$. As U_1 and U_2 are loci of large u, we may expect them to act as high speed arteries and be universal. Similarly, D, a curve of low speed, might be suspected of repelling $\mathbf{x}$ and being a dispersal surface.

The first conjecture is correct as will appear later (Example 7.5.1). That the second is generally false[3] can be gleaned from Problem 6.10.1 (with $a = 0$). Of course, if U_1 and U_2 are both US, a dispersal surface must lie somewhere between them but not necessarily at the bottom of the valley.

Observe how $\mathbf{x}$, if starting from such a point as X_1, might first go to U_1, traverse it downward to its end, then follow a $-45°$ path to U_2, and traverse it to X_2.

Because U_1 and D terminate, there must be changes of sign of u_x on the left part of the plane. These lead us to suspect a transition surface there, on the grounds that P might gain by successive diversions of $\mathbf{x}$ toward regions of higher u.

A later result will show that the universal surfaces are marked by the condition

$$u_x = 0 \tag{7.3.1}$$

but no such general criterion can distinguish dispersal or transition surfaces, for they depend on the $\mathscr{C}$ and the initial conditions.

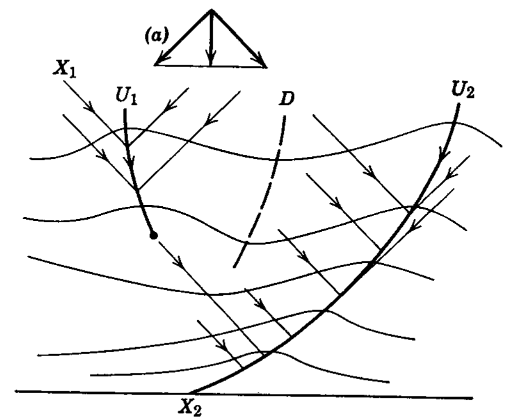


Figure 7.3.5

[3] It is obviously true sometimes, for example, if D is vertical and an axis of symmetry of u.

7.4. THE ANALYTIC NECESSARY CONDITION FOR A LINEAR VECTOGRAM TYPE UNIVERSAL SURFACE

We deal with a one-control variable (and hence a one-player) game. It may and will be assumed to have terminal payoff, as shown in Theorem 2.4.1. At the outset of the preceding section we learned that we must have n (the dimension number) $\geqslant 3$, if a US is to occur.

The KE will be of the form

$$\dot{x}_i = \alpha_i\phi + \beta_i, \qquad i = 1, \ldots n \tag{7.4.1}$$

with α_i and β_i given functions, assumed smooth as need be, of the x_i. The vectors α and β are supposed lineally independent. We may always take for the constraints

$$-1 \leqslant \phi \leqslant 1 \tag{7.4.2}$$

by choosing for β the vector extending to the center of the headline and α the vector reaching from there to one of the ends of the headline.[4]

Suppose we have some definite game, with Value function $V(\mathbf{x})$, which contains a US on which V is supposed smooth and the ME holds. We will have

$$\bar{\phi} = \sigma = -\operatorname{sgn} A \tag{7.4.3}$$

where

$$A = \sum_i \alpha_i V_i$$

whenever $A \neq 0$. Similarly, we denote $\Sigma_i \beta_i V_i$ by B. As the universal surface must be traversed by an intermediate ϕ, say $\breve{\phi}$ with $-1 < \breve{\phi} < 1$, on it we must have

$$A = 0. \tag{7.4.4}$$

Since the ME is $A\bar{\phi} + B = 0$, on the US we must also have

$$B = 0. \tag{7.4.5}$$

Now the tributary paths to the US are obtained by integrating the RPE, utilizing the US and the values of V on it as initial conditions. As there are to be distinct sets of these paths from the two sides of the US, it can only be that $\bar{\phi} = +1$ for one side and -1 for the other. From (7.4.3) then, A is of opposite sign on the two sides.

The RPE of the tributary paths are

$$\mathring{x}_j = -\alpha_j\sigma - \beta_j$$

$$\mathring{V}_j = \sum_i V_i(\alpha_{ij}\sigma + \beta_{ij})$$

where α_{ij} means $\partial\alpha_i/\partial x_j$ etc.

[4] For certain problems, where other constraints are more natural and convenient, our procedure can easily be modified.

Let us see how A changes as we move away from the US on either side.

$$\mathring{A} = \sum_j (\mathring{V}_j\alpha_j + V_j\mathring{\alpha}_j)$$

$$= \sum_j \Big[\sum_i V_i(\alpha_{ij}\sigma + \beta_{ij})\alpha_j + \sum_i V_j\alpha_{ji}(-\alpha_i\sigma - \beta_i)\Big]$$

the latter step utilizing $\mathring{\alpha}_j = \Sigma_i \alpha_{ji}\mathring{x}_i$. If we reverse indices in the latter sum of the bracket, we have

$$\mathring{A} = -\sum_{i,j}(\alpha_{ij}\beta_j - \beta_{ij}\alpha_j)V_i. \tag{7.4.6}$$

But this expression is independent of σ and therefore $\mathring{A}$ is same on both sides of the US. Then it must be zero, for suppose, say, $\mathring{A} > 0$. As $A = 0$ on the US, A would then be positive on both sides at a sufficiently short distance away.

Defining (these quantities are basic)

$$\gamma_i = \sum_j (\alpha_{ij}\beta_j - \beta_{ij}\alpha_j) \tag{7.4.7}$$

the essential part of the sought condition is thus $\sum_i \gamma_i V_i = 0$. We have proved

THEOREM 7.4.1. For a game with one control variable ϕ, terminal payoff, and KE (7.4.1) on a universal surface on and near which V is smooth[5] we must have

$$A = \sum_i \alpha_i V_i = 0$$

$$B = \sum_i \beta_i V_i = 0 \tag{7.4.8}$$

$$C = \sum_i \gamma_i V_i = 0.$$

From the point of view of semipermeable surfaces, we know that the V_i can be considered as the components of the normal vector ν. Thus a purely geometric version of this result runs:

If a region of $\mathscr{E}$, filled by a family of semipermeable surfaces and traversed according to the optimal $\bar{\phi}$ (which forestalls penetration), contains a US, at the US the normal ν to any such surface must satisfy

$$\sum_i \alpha_i\nu_i = \sum_i \beta_i\nu_i = \sum_i \gamma_i\nu_i = 0. \tag{7.4.9}$$

THEOREM 7.4.2. The V_i are continuous across a linear vectogram type universal surface.

Proof. We choose coordinates so that the US lies in the plane: $x_1 = 0$. We suppose a proper US, that is, on it the vectograms do not lie in its

[5] Theorem 7.4.2. renders this hypothesis unnecessary.

plane; in the present coordinates this means that α_1, β_1 cannot both be zero.

The US consists of a family of optimal paths and so V will be known on it. By differention so will $V_2, \ldots, V_n$. Finally, V_1 is determined hereon by at least one of the equations, $A = 0$, $B = 0$, for at least in one the coefficient of V_1 does not vanish.

Now, taking ϕ in turn as ± 1, we can integrate the RPE using initial conditions on the US of which the above values of V_i are part. The result will include the values of V and the V_i as functions over some neighborhood of the US. From the known theory of first-order partial differential equations, and their integration by characteristics, the V_i obtained will be the appropriate partials of V.[6] It is also clear that this V is the Value.

Then, as asserted, the V_i are continuous across the US because at it, from both sides, the V_i approach their unique initial values on the US.

Research Problem 7.4.1. Investigate the continuity of the second partials across a ϕ-US.

There are two benefits of the continuity of V_i at a ϕ-US. One is that the ME is everywhere satisfied. Thus, techniques of proof, such as the verification theorem, apply, and we have the same assurance of the validity of a solution as in a case free of singular surfaces.

The second concerns the control variables other than ϕ. We have stated that, when concerned with a ϕ-US, we can suppose them as already fixed at their optimizing values as dictated by the ME_1. Such values will generally be rather simple functions of the x_i and V_i. The continuity of the latter gives us some assurance that the other control variables will be continuous at the ϕ-US.

These conditions are necessary but, as we shall see, by no means sufficient. Therefore we shall define a candidate for universal surfaces (CUS) as a smooth surface which

C1. satisfies (7.4.9) or (7.4.8).

C2. is the union of a set of paths satisfying

$$\dot{x}_i = \alpha_i \breve{\phi} + \beta_i \qquad \text{for some } \breve{\phi}, \text{ with } -1 < \breve{\phi} < 1.$$

7.5. THE WORKABLE CONDITION WHEN $n = 3$

The conditions (7.4.8) (or (7.4.9)) do not directly instruct us as to how to detect CUS, for they involve the V_i (or ν_i) which are not known at the outset of a problem.

[6] See for example, Reference [6], Ch. 2, §3.2 and 7.

When $n = 3$, however, matters are immediately simplified. As the components of ν are not all 0, on a CUS we must have at once the condition:

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}^{7} = 0 \tag{7.5.1}$$

We have here an equation in the x_i which may be the equation of one or several surfaces. Each such on which C2 also holds is a CUS and conversely.

Our *procedure* will be, whatever n, as follows. When building the solution suppose we encounter a region of $\mathscr{E}$ around which the optimal paths disperse and leave a *void*, that is, a region without paths such as the shaded area of Figure 7.5.1a (when $n = 2$). We seek a CUS. Suppose we find one AB as in (b) within the void. The tributary paths will blend smoothly with those previously found. This is because the linearity of the KE in ϕ imply that those RPE which equate the $\mathring{x}_i$ are free of the V_i. Thus for a definite value choice of σ, a path is determined only from the initial values of the x_i and so paths filling a continuum such as CAB in (b) of the figure will constitute a smooth family.

Thus the void is filled by paths and hence also bears functions $\bar{\phi}$, V, and V_i. What assurance have we that they constitute the correct solution? We have already proved the continuity of V and its partials across the US. This continuity also holds at the junctions such as AF and AG in the figure, for the V_i can be obtained from integrals of the RPE and the initial conditions, even though partly on $\mathscr{C}$ and partly on the US, are

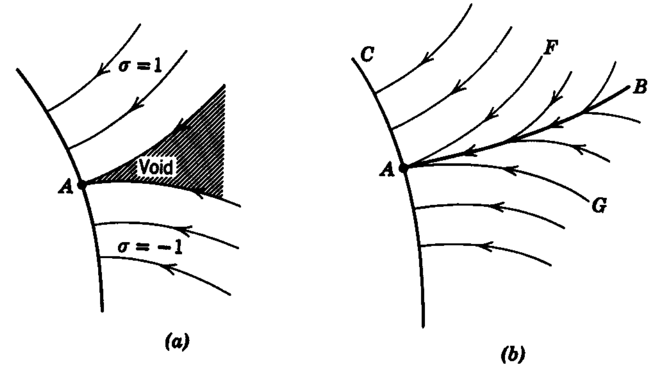


Figure 7.5.1

[7] We have stipulated the linear independence of α and β. Should such fail on a particular surface, this surface may well be a third variety of US, but of a comparatively trivial nature; (7.5.1) will hold for it clearly.

continuous. Thus we have a solution of the ME throughout and techniques, such as the verification theorem, are valid.

Example 7.5.1. A previous illustration. We apply our criterion to the problem depicted in Figure 7.3.5. Changing the notation slightly the KE are

$$\begin{aligned} \dot{x}_1 &= \phi u(x_1, x_2) \\ \dot{x}_2 &= -u(x_1, x_2) \\ \dot{x}_3 &= 1 \qquad\qquad -1 \leqslant \phi \leqslant 1,\ u \geqslant 0. \end{aligned}$$

The third equation arises from our usual transformation of an integral to a terminal payoff (see Section 2.4; here, of course, $G = 1$). Thus we have

$$\alpha_1 = u, \qquad \alpha_2 = \alpha_3 = 0$$

and $\quad \beta_1 = 0, \quad \beta_2 = -u, \quad \beta_3 = 1.$

Recalling $\quad \gamma_i = \sum_j (\alpha_{ij}\beta_j - \beta_{ij}\alpha_j) \qquad$ (7.4.7)

we compute

$$\begin{aligned} \gamma_1 &= \alpha_{12}\beta_2 \quad \text{(all other terms being 0)} \\ &= -u_2 u \quad \left(u_2 = \frac{\partial u}{\partial x_2}\right) \\ \gamma_2 &= -\beta_{21}\alpha_1 = u_1 u \\ \gamma_3 &= 0 \end{aligned}$$

so that the determinant (7.5.1) is

$$\begin{vmatrix} u & 0 & -uu_2 \\ 0 & -u & uu_1 \\ 0 & 1 & 0 \end{vmatrix} = u^2 u_1$$

and as $u > 0$, our condition is (see (7.3.1))

$$u_1 = \frac{\partial u}{\partial x_1} = 0.$$

It clearly holds on U_1, U_2, and D.

Of course, no more is claimed here than the usual type of "necessary conditions" common in minimizing problems: derivative equals zero at minimum of a function; Euler equation at minimum of an integral.

Example 7.5.2. Shortest route of a car, boat, or airplane to a fixed destination. The craft in question moves with a fixed speed, but its curvature is bounded by a given $1/R$. It is steered by choosing at each instant an admissible value of this curvature (corresponding to a steering wheel or

rudder with instantaneous response). We wish to pilot the craft from a given starting position and attitude to within a given distance l of a given destination point A in as short a time as possible. Or, because of the constant speed, we can equivalently speak of the path of least length.

It is clear that we have a special case of the homicidal chauffeur game with an immobile evader. The KE are then the same but with w_2 (E's speed) equal to zero. We replace w_1 by w and also write x_1, x_2 in place of x, y.

The KE, given in Example 2.2.2, then become

$$\dot{x}_1 = -\left(\frac{w}{R}\right) x_2 \phi$$

$$\dot{x}_2 = \left(\frac{w}{R}\right) x_1 \phi - w$$

and, to attain a terminal payoff in the standard way, we adjoin

$$\dot{x}_3 = G = 1.$$

In this 3-space, $\mathscr{C}$ is the cylinder of radius l and axis Ox_3. The standard procedure applied here will yield $\bar{\phi} = \pm 1$ for the right ($x_1 > 0$) and left sides of the upper surface. On any x_3-section of $\mathscr{C}$ the paths will be arcs of concentric circles centered at ($\pm R$, O) as shown in Figure 7.5.2a. Such arcs correspond to sharpest possible right and left rudder, and the paths leave a

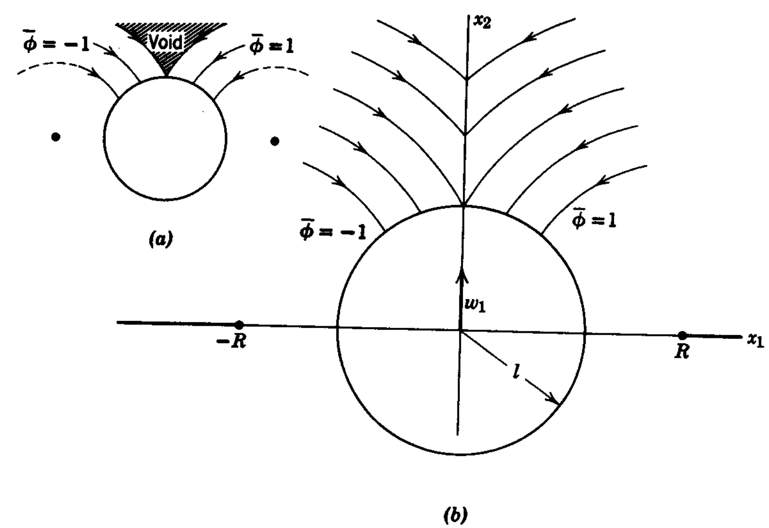


Figure 7.5.2

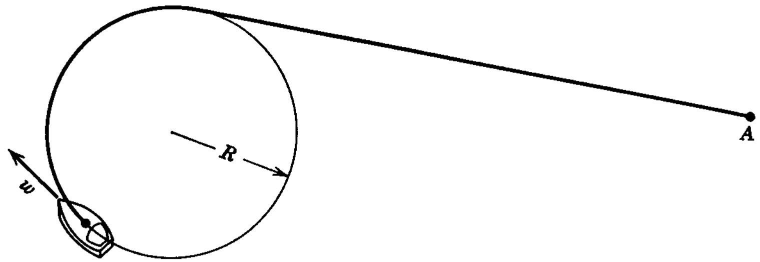


Figure 7.5.3

void (shaded in the figure). Let us apply the CUS criterion. We have

$$\alpha_1 = -\left(\frac{w}{R}\right)x_2, \qquad \beta_1 = 0$$

$$\alpha_2 = \left(\frac{w}{R}\right)x_1, \qquad \beta_2 = -w$$

$$\alpha_3 = 0, \qquad \beta_3 = 1$$

and so, omitting terms which are zero,

$$\gamma_1 = \alpha_{12}\beta_2 = -\left(\frac{w}{R}\right)(-w), \qquad \gamma_2 = \gamma_3 = 0$$

which leads to

$$\begin{vmatrix} -\left(\frac{w}{R}\right)x_2 & 0 & \frac{w^2}{R} \\ \left(\frac{w}{R}\right)x_1 & -w & 0 \\ 0 & 1 & 0 \end{vmatrix} = \left(\frac{w^3}{R^2}\right)x_1 = 0.$$

The plane $x_1 = 0$, being navigable by $\breve{\phi} = 0$, is a CUS. Returning to the two-dimensional version (x_1, x_2-plane with $G = 1$), we use as initial conditions

$$x_1 = 0, \quad x_2 = l + s\,(s \geqslant 0), \quad V = \frac{s}{w}.$$

These are employed with RPE in our usual way, and they lead to tributary paths which are also circular arcs concentric about $(\pm R, 0)$.

The formal details of integrating the RPE, working out V, etc. we leave to the interested reader. Instead we note the simple plausibility of what we have found.

The US, as $\breve{\phi} = 0$, corresponds to straight travel. The tributary paths are sharp turns which precede such a state. In the realistic space, this

means that the craft turns sharply until pointed at the target A and then proceeds there on a straight course,[8] as shown in Figure 7.5.3.

7.6. WHY THE NAME UNIVERSAL SURFACE?

Through problems of the ilk of the preceding. One is startled at first, when working with the ME relevant to, say, steering a boat, to learn that the formal mathematics would seem to dictate always sharp right or left rudder. One knows better; boats are not piloted this way. On a long (ideal) voyage on a uniform sea, on the other hand, the rudder should almost always be straight.

In the last example, the US is a set of measure zero in the reduced space $\mathscr{E}$. Yet it alone marks straight travel and, despite its nullity of area, it is the bearer of all long voyages save brief initial maneuvers.

The possibly pretentious name "universal surface" was intended to direct attention to the importance of these loci.

7.7. THE CALCULUS OF VARIATIONS VIEWPOINT

The universality of US is further clarified as their connection to the Euler equation becomes apparent. We adhere to the three-dimensional case here and will resume this aspect of the subject in a later section after the required additional analysis.

Let us suppose we have a game with integral payoff in the plane with G linear in ϕ. That is, the KE are

$$\dot{x} = \dot{x}_1 = \alpha_1\phi + \beta_1$$

$$\dot{y} = \dot{x}_2 = \alpha_2\phi + \beta_2 \qquad (\alpha_i, \beta_i = \text{functions of } x, y \text{ only}) \qquad (7.7.1)$$

and $\quad G = \alpha_3\phi + \beta_3.$

As was done earlier (Section 5.3), we can cast all in the calculus of variations mold. Writing

$$y' = \frac{dy}{dx} = \frac{\alpha_2\phi + \beta_2}{\alpha_1\phi + \beta_1}$$

we solve for ϕ and substitute the result in

$$\int \frac{\alpha_3\phi + \beta_3}{\alpha_1\phi + \beta_1}\,dx. \qquad (7.7.2)$$

[8] We have not solved this problem completely, of course. There are interesting possibilities when A is initially within one of the minimal turn circles. See the homicidal chauffeur problem, of which, as remarked, this is a special case.

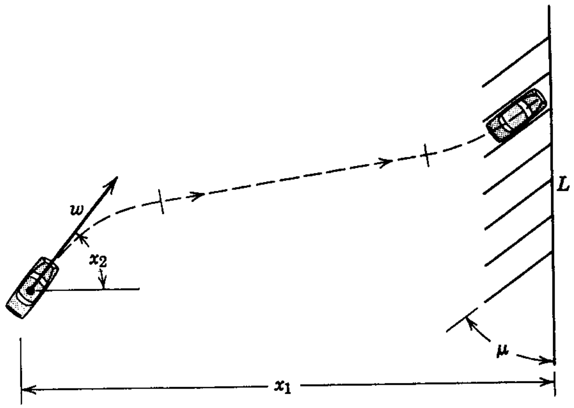


Figure 7.7.1

We have reverted to the classic minimization of an integral of the type

$$\int F(x, y, y')\, dx. \tag{7.7.3}$$

The Euler equation of this integral turns out to be of zero order. That is, it is not a differential equation at all but agrees with the US condition (7.5.1).

Exercise 7.7.1. Carry out the preceding operations in detail and obtain the F of (7.7.3). Show that the classical Euler equation

$$\frac{d}{dx} F_{y'} - F_y = 0$$

here is

$$\left(\frac{\alpha_1\beta_3 - \alpha_3\beta_1}{\alpha_1\beta_2 - \alpha_2\beta_1}\right)_x + \left(\frac{\alpha_2\beta_3 - \alpha_3\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1}\right)_y = 0 \tag{7.7.4}$$

and that this equation is equivalent to (7.5.1) when the α_i, β_i are free of x_3.[9]

The role of our analytic condition for US is now clearer. It is subject to the same intracies of "necessary and sufficient condition" type as fill so many pages of most calculus of variations texts. In Section 7.12 this idea will be discussed further.

As an instance of how (7.5.1) may facilitate calculus of variations problems, we consider

Example 7.7.1. Parking a car. The car moves as in Example 7.5.2. It is located in the large unobstructed parking lot of Figure 7.7.1 and is to park

[9] Recall that we postulated linear independence of α and β and therefore the denominators in (7.7.4) $\neq 0$.

at the line L, its terminal inclination of L being the given angle μ. It is to do so in the least time.

We can confidently expect that, if the initial distance from L is at all great, that most of the optimal path will be straight. The novelty here over Example 7.5.2 is that final as well as preliminary turning maneuvers are required, the former to attain the angle μ. We confine our interest to the following question: Should the straight part of the optimal path be perpendicular to L or, in anticipation of the final turn, should it slope a little μ-wise (such as the dashed path of the figure)?

We use the reduced coordinates x_1, x_2 as shown in the figure. The KE are then

$$\dot{x}_1 = -w \cos x_2$$

$$\dot{x}_2 = -\left(\frac{w}{R}\right)\phi$$

and, as $G = 1$, we adjoin

$$\dot{x}_3 = 1.$$

Calculating the γ_i we obtain the table

i	α_i	β_i	γ_i
1	0	$-w \cos x_2$	$(w^2/R) \sin x_2$
2	$-(w/R)$	0	0
3	0	1	0

The condition (7.5.1) then is

$$-\left(\frac{w^3}{R^2}\right) \sin x_2 = 0.$$

Any horizontal line is then a CUS. Our question appears to be answered in favor of the former alternative: aside from the initial and final turns, the car approaches L perpendicularly.

7.8. ALL STRATEGIES OPTIMAL

There may be regions in $\mathscr{E}$ in which any value of ϕ serves to achieve the Value of the game. We will limit ourselves to games of the type describable by (7.7.1).

The main result will show that a condition is the identical satisfaction of our CUS condition.

The ME is

$$\min_{\phi} [\alpha_1 V_x + \alpha_2 V_y + \alpha_3]\phi + [\beta_1 V_x + \beta_2 V_y + \beta_3] = 0. \tag{7.8.1}$$

If in a region $\mathscr{R} \subset \mathscr{E}$, all ϕ are optimal, then both brackets equal zer there. We have simultaneous equations for V_x and V_y, and their solution i

$$V_x = \frac{\alpha_2\beta_3 - \alpha_3\beta_2}{\alpha_1\beta_2 - \alpha_2\beta_1} = M, \qquad V_y = \frac{\alpha_3\beta_1 - \alpha_1\beta_3}{\alpha_1\beta_2 - \alpha_2\beta_1} = -N. \quad (7.8.2)$$

Hence we must have identically ($V_{xy} = V_{yx}$)

$$M_y + N_x = 0 \quad (7.8.3)$$

but this is the form (7.7.4) of the US condition.

Conversely if this condition holds in a region $\mathscr{R}$, then there must exist there a *function* $V(x, y)$ satisfying (7.8.2) and hence annulling both the brackets of (7.8.1) and hence yielding identical satisfaction of the ME. But this function will be the Value of the game only if its partials jibe with those of the known Value on the boundary of $\mathscr{R}$.

Example 7.8.1. The vectograms are all of type shown in Figure 7.8.1; $G = 1$ or payoff = time to termination. That is, the KE are

$$\dot{x} = \phi, \qquad \dot{y} = -1, \qquad -1 \leqslant \phi \leqslant 1.$$

If $\mathscr{C}$ is as shown, then it is immediate that

in ① any ϕ is optimal
in ② $\bar{\phi} = 1$
in ③ $\bar{\phi} = -1$
in ④ termination is impossible.

Yet the condition (7.8.3) holds everywhere, for all α_i, β_i are constant.

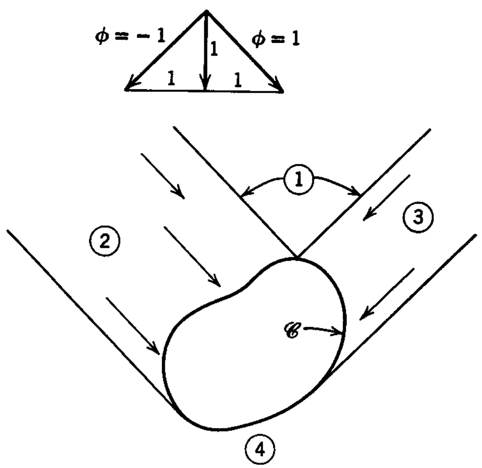


Figure 7.8.1

Research Problem 7.8.1. Investigate the relation between the lack of optimal paths and the more general CUS conditions of this chapter.

7.9. THE WORKABLE CRITERION WHEN $n = 4$

When n, the dimension of $\mathscr{E}$ or number of state variables, exceeds 3, the basic conditions

$$\Sigma_i \alpha_i \nu_i = \Sigma_i \beta_i \nu_i = \Sigma_i \gamma_i \nu_i = 0 \quad (7.4.9)$$

where ν is the vector normal to surfaces of constant V at the CUS, no longer determines a CUS directly. In this section we shall exhibit, when $n = 4$, a technique, entailing differential equations, for locating such surfaces. This is as far as we shall go with generality, but subsequent examples will display a few sporadic instances where we construct US with $n > 4$.

The formal result will of the type

$$Q\phi + R = 0 \quad (7.9.1)$$

where Q and R are functions of $x_1, \ldots, x_4$. Thus if $Q = 0$, the equation $R = 0$ will be that of a possible US. But if $Q \neq 0$, (7.9.1) will determine ϕ as $\check{\phi}(\mathbf{x})$, a function which when inserted into the KE

$$\dot{x}_i = \alpha_i \phi + \beta_i, \qquad i = 1, 2, 3, 4\text{[10]} \quad (7.4.1)$$

will render them differential equations. Since the totality of their integrals can be expected to fill more than a surface, appropriate initial conditions must be selected. It is likely that they will be supplied naturally by the problem at hand.

Let a game with one control variable, ϕ, and terminal payoff have a US. An *artery* will be defined as the intersection of the US and a surface of constant V. Thus an artery is a "curve," that is, it has dimension $n - 2$. The surface of constant V containing it consists of the artery and the two sets of tributary paths (those with $\bar{\phi} = +1$ and -1) feeding into it.

Let a surface $\mathscr{S}$ of constant V be given by

$$x_i = X_i(s, t, \tau) \quad (7.9.2)$$

where s, t, τ are parameters enjoying the following special properties:

The retrograde time parameter, as usual, on the tributary paths is τ. The artery of $\mathscr{S}$ is given by

$$x_i = X_i(s, t, 0). \quad (7.9.3)$$

[10] In this section such indices will hereafter be understood to run from 1 to 4 without specific mention.

Navigation in the US is accomplished with $\phi = \check{\phi}$ and the time parameter is t. That is, the artery consists of the paths (7.9.2) where s is held constant and τ held 0; thus when $\tau = 0$

$$X_{it}\left(= \frac{\partial X_i}{\partial t}\right) = \alpha_i\check{\phi} + \beta_i. \tag{7.9.4}$$

The normal ν to $\mathscr{S}$ must satisfy (7.4.9). This means that, up to an arbitrary multiplier which is a function of the x_i, the ν_i are the 3-minors (with the appropriate alternating signs) of the 4×3 matrix

$$(\alpha_i, \beta_i, \gamma_i). \tag{7.9.5}$$

By choosing such a determination, we suppose the ν_i are known functions of the x_i.

As the ν_i are normal to $\mathscr{S}$

$$\Sigma_i \nu_i X_{iw} = 0 \tag{7.9.6}$$

where

$$X_{iw} = \frac{\partial X_i}{\partial w} \quad \text{and} \quad w = s, t, \text{or } \tau.$$

If $w = s$ or τ, differentiating (7.9.4) with respect to w gives

$$X_{itw} = \sum_j (\alpha_{ij}\check{\phi} + \beta_{ij})X_{jw} + \alpha_i\check{\phi}_w.$$

We also differentiate (7.9.6) with respect to t and use the above for X_{iwt}:

$$\sum_i X_{iw}\sum_j \nu_{ij}(\alpha_j\check{\phi} + \beta_j) + \sum_{i,j} \nu_i(\alpha_{ij}\check{\phi} + \beta_{ij})X_{jw} = 0. \tag{7.9.7}$$

Note that $\check{\phi}_w$ does not appear as

$$\sum_i \alpha_i\nu_i = 0.$$

The latter equality also implies, on differentiating with respect to x_j,

$$\sum_j \nu_i\alpha_{ij} = -\sum_i \alpha_i\nu_{ij} \tag{7.9.8}$$

and, similarly, when β replaces α. We make these substitutions in (7.9.7) and reverse indices in the second sum, leading to

$$\sum_i X_{iw}\Big[\sum_j (\nu_{ij} - \nu_{ji})(\alpha_j\check{\phi} + \beta_j)\Big] = 0. \tag{7.9.9}$$

Define $\Delta_{ij} = \nu_{ij} - \nu_{ji}$.

Observe that (7.9.9) holds also when $w = t$, for then it is

$$\sum_{i,j} \Delta_{ij}X_{it}X_{jt} = 0$$

which is true, for Δ_{ij} is a skewsymmetric matrix.

Now, if $\mathscr{S}$ is to be a nondegenerate surface, one of the three minors of the "Jacobian matrix,"

$$\left(\frac{\partial X_i}{\partial w}\right), \quad w = s, t, \tau$$

is not zero. Let it be the one obtained by deleting the column where $i = k$. Then, multiplying (7.9.6) by

$$\sum_j \Delta_{kj}(\alpha_j\check{\phi} + \beta_j)$$

and (7.9.9) by ν_k and subtracting, we obtain

$$\sum_i{}' \Big[\sum_j (\nu_i\Delta_{kj} - \nu_k\Delta_{ij})(\alpha_j\check{\phi} + \beta_j)\Big]X_{iw} = 0 \tag{7.9.10}$$

where the prime on the sum signifies the deletion of the index value k. Because the determinant $|X_{iw}|_{i \neq k} \neq 0$, we must have satisfied the condition we will call

$$C_{ik}: \quad \sum_j (\nu_i\Delta_{kj} - \nu_k\Delta_{ij})(\alpha_j\check{\phi} + \beta_j) = 0. \tag{7.9.11}$$

This is the condition (7.9.1) promised earlier.

Actually all the conditions C_{ik} (for $i \neq k$) turn out to be equivalent. This follows from the *purely algebraic* lemma below. To apply it we must first show that

$$\sum_{i,j} \Delta_{ij}\alpha_i\beta_j = 0. \tag{7.9.12}$$

This follows from

$$\sum_i \nu_i\gamma_i = 0 = \sum_{i,j} \nu_i(\alpha_{ij}\beta_j - \beta_{ij}\alpha_j)$$

and we make replacements of type (7.9.8), which remove the α_{ij} and β_{ij}, and then reverse indices is one of the sums.

LEMMA 7.9.1. Let α, β, ν be vectors in 4-space such that

$$\sum_i \alpha_i\nu_i = \sum_i \beta_i\nu_i = 0 \tag{7.9.13}$$

and α and β are lineally independent. Let $\Delta = (\Delta_{ij})$ be a skewsymmetric 4-matrix such that

$$\sum_{i,j} \Delta_{ij}\alpha_i\beta_j = 0. \tag{7.9.14}$$

Define

$$\left.\begin{matrix}A_{uv}\\ B_{uv}\end{matrix}\right\} = \sum_j \left.\begin{matrix}\alpha_j\\ \beta_j\end{matrix}\right\}(\nu_u \Delta_{vj} - \nu_v \Delta_{uj}). \tag{7.9.15}$$

Then for any two pairs of indices (u, v), the pairs A_{uv}, B_{uv} are lineally dependent. That is, either all $A_{uv} = 0$ or else some do not and the ratio B_{uv}/A_{uv} is the same for all such, while $B_{uv} = 0$ for the rest.

Proof. Let us perform a rotation with matrix $R = (r_{ij})$ in the 4-space containing α, β, and ν. Thus if $\bar{\alpha}$, $\bar{\beta}$, $\bar{\nu}$ represent these vectors in the new coordinates, we will have $\alpha = R\bar{\alpha}$, $\beta = R\bar{\beta}$, $\nu = R\bar{\nu}$ where the first relation, say, means

$$\alpha_i = \sum_j r_{ij}\bar{\alpha}_j.$$

The matrix itself satisfies $RR' = E$ or $\sum_u r_{ui}r_{uj} = \sum_u r_{iu}r_{ju} = \delta_{ij}$. The equivalent of Δ in the new coordinates is $\bar{\Delta}$, where $\Delta = R\,\bar{\Delta}R'$ or, when written out,

$$\Delta_{ij} = \sum_{s,t} \bar{\Delta}_{st} r_{is} r_{jt}.$$

One can verify standardly and easily that

$$\sum_{ij} \bar{\Delta}_{ij}\bar{\alpha}_i\bar{\beta}_j = \sum_{i,j} \Delta_{ij}\alpha_i\beta_j$$

and that $\bar{\Delta}$ is skewsymmetric.

We next show that the skewsymmetric matrices A and B transform in the normal way, that is, as Δ does.

Substituting into (7.9.15)

$$A_{uv} = \sum_{j,p,q,s,t} (r_{jp}\bar{\alpha}_p)(r_{uq}\bar{\nu}_q)(\bar{\Delta}_{st} r_{vs} r_{jt}) - \cdots$$

where the dots stand for the same expression with u and v interchanged. Summing over j gives

$$\begin{aligned} A_{uv} &= \sum_{q,s,t} \bar{\alpha}_t \bar{\nu}_q\, \bar{\Delta}_{st}(r_{uq}r_{vs} - r_{vq}r_{us}) \\ &= \sum_{q,s,t} \bar{\alpha}_t(\bar{\nu}_q\, \bar{\Delta}_{st} - \bar{\nu}_s\, \bar{\Delta}_{qt}) r_{uq} r_{vs} \\ &= \sum_{q,s} \bar{A}_{qs} r_{uq} r_{vs} \end{aligned} \tag{7.9.16}$$

and a similar relation holds for B, β's replacing α being the sole change in the proof.

If $\nu = 0$, the lemma is true trivially as all A_{uv} and $B_{uv} = 0$. If $\nu \neq 0$, we can now select the rotation R. Choose the first axis along ν. The second is chosen along α, which is allowed by (7.9.13). Similarly, we can

choose the third in the plane of α and β. Then in these new coordinates

$$\begin{aligned} \bar{\nu} &= (\bar{\nu}_1, 0, 0, 0) \\ \bar{\alpha} &= (0, \bar{\alpha}_2, 0, 0) \\ \bar{\beta} &= (0, \bar{\beta}_2, \bar{\beta}_3, 0) \end{aligned}$$

with $\bar{\nu}_1, \bar{\alpha}_2, \bar{\beta}_3 \neq 0$, the latter two following from the independence of α and β. Now (7.9.14) is

$$\bar{\Delta}_{23}\bar{\alpha}_2\bar{\beta}_3 = 0$$

implying

$$\bar{\Delta}_{23} = \bar{\Delta}_{32} = 0.$$

Note that $\bar{A}_{uv} = 0$ when $u \neq 1$ and $v \neq 1$ and, from the skewsymmetry, $\bar{A}_{11} = 0$. In the remaining cases we have (let us take $\bar{\nu}_1 = 1$)

$$\bar{A}_{1v} = -\bar{A}_{v1} = \bar{\alpha}_2\, \bar{\Delta}_{v2}$$

so that

$$\begin{aligned} \bar{A}_{12} &= 0 \\ \bar{A}_{13} &= 0 \\ \bar{A}_{14} &= \bar{\alpha}_2\, \bar{\Delta}_{42}. \end{aligned}$$

Similarly,

$$\bar{B}_{1v} = -\bar{B}_{v1} = \bar{\beta}_2\, \bar{\Delta}_{v2} + \bar{\beta}_3\, \bar{\Delta}_{v3}$$

and so

$$\begin{aligned} \bar{B}_{12} &= 0 \\ \bar{B}_{13} &= 0 \\ \bar{B}_{14} &= \bar{\beta}_2\, \bar{\Delta}_{42} + \bar{\beta}_3\, \bar{\Delta}_{43}. \end{aligned}$$

Thus from (7.9.17)

$$A_{uv} = \bar{A}_{14}(r_{u1}r_{v4} - r_{u4}r_{v1});$$

similarly,

$$B_{uv} = \bar{B}_{14} \times \text{(the same factor)}$$

and the lemma is proved.

To summarize our conclusions let us put

$$U^j_{uv} = \nu_u\, \Delta_{vj} - \nu_v\, \Delta_{uj}$$

where as before

$$\Delta_{ij} = \nu_{ij} - \nu_{ji}.$$

We have established

THEOREM 7.9.1. In a game with one control variable, ϕ, terminal payoff, linear vectograms, and $n = 4$, a CUS must be a surface on which the equivalent conditions C_{uv} $(u \neq v)$ hold and not all the $\nu_j \neq 0$, where C_{uv} means

$$\Big(\sum_j \alpha_j U^j_{uv}\Big)\breve{\phi} + \Big(\sum_j \beta_j U^j_{uv}\Big) = 0. \tag{7.9.17}$$

A first example is in the form of

Exercise 7.9.1. For the KE

$$\dot{x}_1 = \left(\frac{x_1x_3}{x_2}\log x_1\right)\phi + x_2$$
$$\dot{x}_2 = x_3\phi$$
$$\dot{x}_3 = x_1\phi$$
$$\dot{x}_4 = x_4$$

carry through the computations and ascertain that the equations C_{uv} consist of

$$(x_1x_3 \log x_1)\breve{\phi} - x_2^2$$

equated to zero after being multiplied by the factors in each case:

$$C_{12}:\ -x_4^2 \qquad C_{23}:\ -x_2x_3\log x_1$$
$$C_{13}:\ x_1/x_3 \qquad C_{24}:\ -x_2^3/x_4$$
$$C_{14}:\ 0 \qquad C_{34}:\ x_1/x_3.$$

To obviate the ambiguity in the determination of the ν_i take $\nu_1 = x_2x_4$.

Problem 7.9.1. Note that in the previous exercise, x_4 appears only in the fourth KE. Such is tendentious of the first three equations forming an independent system. Treat this trio by the method of Section 7.5 and obtain the CUS. Note that such satisfy the differential equations of Exercise 7.9.1, that is, the KE with $\breve{\phi}$ replacing ϕ, but do not seem part of the formal general solution. Are they singular solutions?

The ν_i in our work are of arbitrary length. If they are all multiplied by an arbitrary function, the intermediate steps will be altered; for example, there will be terms added to the Δ_{ij}. But that the final result is unaffected we leave the reader to show in

Problem 7.9.2. If the ν_i are replaced by $h\nu_i$, where h is a differentiable function of $x_1, \ldots x_4$, show that the C_{uv} each become multiplied by h^2.

Research Problem 7.9.1. Our technique entails essentially the six equations C_{uv}, which we proved equivalent. One suspects that there should be a more direct approach eschewing this polychotomy. Is there?

Example 7.9.1. The classical brachistochrone with bounded curvature. Let us once again return to our vehicle with bounded curvature but let the speed now be a given function of x and y. Then the reduction of coordinates in Examples 7.5.2 and 7.7.1 is no longer possible.

Let us take x_1 and x_2 (x and y) as coordinates in the plane. If we let the

speed be $\sqrt{x_2}$ (see Example 5.2) we have, aside from the curvature constraint, the classical calculus of variations brachistochrone problem. If x_3 is the inclination of the velocity vector of the x_1-axis, the KE are

$$\dot{x}_1 = \sqrt{x_2}\cos x_3$$
$$\dot{x}_2 = \sqrt{x_2}\sin x_3$$
$$\dot{x}_3 = -(\sqrt{x_2}/R)\phi, \qquad -1 \leqslant \phi \leqslant 1$$
$$\dot{x}_4 = 1.$$

Tabulating α_i, β_i, and γ_i, after computing the latter from (7 4.7):

$$\gamma_i = \sum_j (\alpha_{ij}\beta_j - \beta_{ij}\alpha_j)$$

i	α_i	β_i	γ_i
1	0	$\sqrt{x_2}\cos x_3$	$(x_2/R)\sin x_3$
2	0	$\sqrt{x_2}\sin x_3$	$-(x_2/R)\cos x_3$
3	$-\sqrt{x_2}/R$	0	$(\frac{1}{2}R)\sin x_3$
4	0	1	0

For ν we use

$$\nu_1 = -\cos x_3, \quad \nu_2 = -\sin x_3, \quad \nu_3 = 0, \quad \nu_4 = \sqrt{x_2}$$

which is seen to be orthogonal to α, β, γ. After ascertaining the Δ_{ij} and then the $U_{12}{}^i$ we find for C_{12}

$$-\left(\frac{\sqrt{x_2}}{R}\right)\breve{\phi} + \frac{1}{2\sqrt{x_2}}\cos x_3 = 0.$$

[As a check, C_{24} turns out to be

$$-\left(\frac{x_2}{R}\right)\breve{\phi}\cos x_3 - \frac{1}{2}\sin^2 x_3 + \frac{1}{2} = 0,$$

the same thing.]

Thus $\breve{\phi} = R\cos x_3/2x_2$ and we are led to the differential equations for the artery:

$$\dot{x}_1 = \sqrt{x_2}\cos x_3$$
$$\dot{x}_2 = \sqrt{x_2}\sin x_3$$
$$\dot{x}_3 = -\cos x_3/2\sqrt{x_2}$$
$$\dot{x}_4 = 1.$$

The general solution of the first three is, as may readily be checked,

$$x_1 = c_1 + K(\theta - \sin\theta)$$
$$x_2 = K(1 - \cos\theta)$$
$$x_3 = \tfrac{1}{2}(\pi - \theta)$$

where $\theta = (t + c_3)/\sqrt{2K}$ and c_1, c_3, K are constants.

Note that the first two are the equations of the classical cycloids. We have here an instance of a general principle which will be elucidated in Section 7.12.

Exercise 7.9.2. A simplification of the last example. The point **x** moves in the plane with *constant* speed w but with curvature bounded by $1/R$. The payoff is the time to reach a given curve $\mathscr{C}$. The arbitrariness of $\mathscr{C}$ precludes reducing the coordinates to two, so that this problem must be treated in the manner above.

1. Show that CUS are straight lines.
2. Show, by taking into account the initial conditions, that a further condition is that these lines be perpendicular to $\mathscr{C}$. The ideas of Section 7.10 will be helpful. (Similarly, one can attain this same perpendicularity condition for the above cycloids.)

Example 7.9.2. The Battle of Bunker Hill. This problem belongs to a later chapter. Here we confine our attention to the computation of a US. We need but know the KE, which are

$$\dot{x}_1 = -c_2x_2p_2(x_4)$$
$$\dot{x}_2 = -c_1x_1p_1(x_4)\phi$$
$$\dot{x}_3 = -c_1\phi$$
$$\dot{x}_4 = -1 \qquad (7.9.18)$$

with $0 \leqslant \phi \leqslant 1$; p_1 and p_2 are given functions of x_4; c_1 and c_2 are constants.

Writing the α_i and β_i in the following table and then completing it by computing the γ_i we obtain

i	α_i	β_i	ν_i
1	0	$-cx_2p_2$	$-c_1c_2x_1p_1p_2$
2	$-c_1x_1p_1$	0	$c_1[c_2x_2p_1p_2 + x_1p_1']$
3	$-c_1$	0	0
4	0	-1	0.

The simplest ν_i appears to be

$$\nu_1 = c_2x_2p_1p_2 + x_1p_1'$$
$$\nu_2 = c_2x_1p_1p_2$$
$$\nu_3 = -c_2x_1^2p_1^2p_2$$
$$\nu_4 = -c_2x_2p_2(c_2x_2p_1p_2 + x_1p_1').$$

C_{12}, say, leads to an apparently formidable result:

$$\breve{\phi} = \frac{1}{2c_1c_2p_1^3p_2^2}[2c_2^2x_2^2p_1^2p_2^3 + 2c_2x_1x_2p_1p_2^2p_1' + x_1^2(p_1p_1'p_2' + 2p_2p_1'^2 - p_1p_2p_1'')].$$

But if we put

$$u = \frac{x_2}{x_1}, \qquad W = \frac{p_1'}{p_1^2p_2}$$

things appear more reasonable:

$$\breve{\phi} = \frac{1}{c_1}Z \qquad (7.9.19)$$

where

$$Z = c_2u^2\frac{p_2}{p_1} + u\frac{p_1'}{p_1^2} - \frac{W'}{2c_2}.$$

The differential equations for the artery in retrograde form, become

$$\mathring{x}_1 = c_2x_2p_2$$
$$\mathring{x}_2 = x_1p_1Z$$
$$\mathring{x}_3 = Z$$
$$\mathring{x}_4 = 1.$$

We are going to integrate with the initial conditions

$$x_i = s_i(i = 1, 2, 3), \qquad s_4 = 0.$$

Thus we will have $x_4 = \tau$ and to the ° and ′ derivatives are the same. Now

$$\mathring{u} = \left(\frac{\mathring{x}_2}{x_1}\right) = \frac{x_1p_1Z}{x_1} - \left(\frac{x_2}{x_1^2}\right)c_2x_2p_2$$
$$= p_1\left[Z - \frac{cu^2p_1}{p_2}\right] = \frac{up_1'}{p_1} - \frac{p_1W'}{2c}$$

and this equation integrates easily to

$$u = p_1\left(K - \frac{W}{2c_2}\right). \qquad (7.9.20)$$

Here K is the integration constant and is easily seen to be

$$K = \left(\frac{s_2}{s_1}\right)\frac{1}{p_1(0)} + \frac{W(0)}{2c_2}. \tag{7.9.21}$$

Putting $Q(\tau) = \int_0^\tau p_1(x)p_2(x)\,dx$, we easily perform the full integration:

$$x_1 = \frac{s_1}{\sqrt{p_1}}\, e^{c_2 KQ}$$

$$x_2 = ux_1 \text{ (use the above values of } u \text{ and } x_1) \tag{7.9.22}$$

$$x_3 = cK^2Q(\tau) - \frac{1}{4c_2}\int_0^\tau \frac{p_1'^2}{p_1^3 p_2}\,d\tau - \frac{W(\tau) - W(0)}{2c_2} + s_3$$

$$x_4 = \tau.$$

Example 7.9.3. The war of attrition; second version. Again this is a game whose significance will be discussed later; here only the formal analysis of its US will interest us. It entails a short cut.

This time it turns out that the KE lead us to the table:

i	α_i	β_i	γ_i
1	0	m_1	0
2	$-cx_1x_2$	m_2	$-c_2(m_1x_2 + m_2x_1)$
3	0	-1	0
4	x_1	$x_2 - x_1$	$m_1 + c_2x_1x_2$

Here c_2, m_1, m_2, are constants. As usual, the first two columns are obtained from the KE and the γ_i are then computed.

From the first and third columns we observe at once a result of the form

$$\alpha_2\nu_2 + \alpha_4\nu_4 = 0$$

$$\gamma_2\nu_2 + \gamma_4\nu_4 = 0.$$

Then either the determinant here is zero or $\nu_2 = \nu_4 = 0$.

The latter alternative requires also that, say, $\nu_1 = 1$, $\nu_3 = m_1$. As constant ν_i lead to null Δ_{ij} our general criterion is trivially satisfied by all surfaces. These values of the normals themselves, pertaining only to the surfaces of constant V, tell us nothing.

As for the first alternative:

$$\begin{vmatrix} x_1x_2 & m_1x_2 + m_2x_1 \\ x_1 & m_1 + c_2x_1x_2 \end{vmatrix} = x_1^2(c_2x_2^2 - m_2) = 0.$$

The problem to which we will apply these results outlaws $x_1 = 0$ so that we are left with the surface

$$x_2 = \sqrt{m_2/c_2}. \tag{7.9.23}$$

As will be seen later, this just fits the requirements.

7.10. A TEST FOR A VOID AND A FURTHER NECESSARY CONDITION FOR A UNIVERSAL SURFACE

We continue with games having KE (7.4.1). The initial conditions are given on a surface $\mathscr{C}$ which may be the terminal surface or some other that arises during solution to play a similar role. Suppose there is a curve $\mathscr{K}$ on $\mathscr{C}$ at which $\bar\phi$ changes value, that is, A must change sign at $\mathscr{K}$, if not on $\mathscr{C}$, then at least in $\mathscr{E}$ near $\mathscr{C}$.[12]

There will be a family of paths emanating from $\mathscr{C}$ on each side of $\mathscr{K}$ and, generally, one of two things will occur. Either the two families will mutually intersect and thereby induce a dispersal surface, as was studied in Chapter 6 (see Figure 6.5.1), or they will diverge and leave a void, free of paths between them, indicative of a universal surface (see Figure 7.5.1).[13] We seek here a criterion to distinguish these two cases.

We define

$$D = \sum_{i,j} (\gamma_{ij}\alpha_j - \alpha_{ij}\gamma_j)V_i \tag{7.10.1}$$

where $\gamma_{ij} = \partial\gamma_i/\partial x_j$.

THEOREM 7.10.1. Under the above circumstances, if V exists in $\mathscr{E}$ near $\mathscr{C}$ and has continuous partials, and the paths leave a void, then at $\mathscr{K}$

$$D \leqslant 0. \tag{7.10.2}$$

Proof. In $\mathscr{E}$ near $\mathscr{C}$, A is a continuous function which changes sign at some surface passing through $\mathscr{K}$ and so is decreasing as $\mathbf{x}$ passes through this surface in the direction from the side where $\bar\phi = -1$ to that where $\bar\phi = +1$. If the paths from these two sides leave a void, then, at $\mathscr{K}$, A must increase more rapidly with τ on the former side than on the latter.

These rates of increase cannot be compared through the use of $\mathring{A}$, for (7.4.6) evaluates this quantity here also and shows it independent of ϕ.

We must look to $\mathring{\mathring{A}}$ which, after a calculation along the same lines, turns out to be $D\phi + E$, where D is given (7.10.1) and E is an expression of similar structure. As $\mathring{\mathring{A}}$ must be greater (or at least equal) when $\bar\phi = -1$ than when $\bar\phi = +1$, we have (7.10.2).

[12] In Example 7.5.1, if $\mathscr{C}$ is taken as $y = 0$, then one can verify that $A = 0$ and $\bar\phi$ is determined by sgn $\mathring{A}$.

[13] Of course, an intermediate case is also possible.

COROLLARY 7.10.1. The condition (7.10.2) must hold everywhere on a ϕ-US with linear vectograms.

For we may cut the US with an arbitrary smooth surface and, using the values of V on it as initial conditions, let it play the role of $\mathscr{C}$ above, because the solution on the far side will agree with the original. The intersection with the US will be $\mathscr{K}$; (7.10.2) holds there; hence at all of the US.

Of course, should $D = 0$, we could proceed with higher derivatives of A. But we shall not pursue this subject. It is beset with the classical tribulations of attempting sharply to prescribe conditions for a minimum in terms of derivatives. When working individual problems, we can take $D < 0$ as a strong criterion for a US and expect that its negation, $D > 0$, on $\mathscr{C}$ will indicate a dispersal surface. A typical instance is

Exercise 7.10.1 Show that in Example 7.5.1

$$D = uu_{xx}$$

so that ridges in Figure 7.3.5 are eligible to be US but not the valleys.

7.11. TEST FOR A TRANSITION SURFACE

Although this topic in substance is not germane to the present chapter, its formal aspect is related.

Suppose that, when dealing with the integrals of the RPE, in some stage of the solution of a game of the type of the preceding sections, we encounter a surface $\mathscr{S}$ on which $A = \sum_i \alpha_i V_i = 0$. That is, on each path, we find a certain value of τ such $A(\tau) = 0$, and the totality of the points at which this occurs comprises our surface.

Under what conditions are we justified in asserting that $\mathscr{S}$ is a transition surface, that is, one at which $\bar{\phi}$ changes from one of its extreme values to the other?

As we proceed along an optimal path, by the same reasoning as earlier,

$$\mathring{A} = -\sum_i \gamma_i V_i$$

and is independent of ϕ. Suppose at $\mathscr{S}$, $\mathring{A} \neq 0$. As $A = 0$ there, then A certainly changes sign upon crossing $\mathscr{S}$. Therefore it cannot be that $\bar{\phi}$ remains unchanged, and so $\mathscr{S}$ is a transition surface. Thus we can state

THEOREM 7.11.1. When the integration of the RPE leads to a surface crossed by the paths and at which

$$\sum_i \alpha_i V_i = \sum_i \beta_i V_i = 0, \qquad \sum_i \gamma_i V_i \neq 0$$

then this surface is a transition surface.

The utility of this result: Suppose that we have analyzed a game according to the ideas of this chapter and have detected all surfaces at which

$$\sum_i \alpha_i V_i = \sum_i \beta_i V_i = \sum_i \gamma_i V_i = 0.$$

Then should we encounter as above a surface on which $A = 0$ but which does not belong to this class, we know at once it is a transition surface.

7.12. FURTHER DISCUSSION OF THE BASIC NATURE OF UNIVERSAL SURFACES AND THEIR RELATION TO THE EULER EQUATION

We will erect our ideas around a generalization of Example 7.9.1 and Exercise 7.9.2, but will adopt a broader point of view. We consider first the general.

Problem I. The point $\mathbf{x}$ moves in the plane (coordinates $= x_1, x_2$) with a speed $w(x_1, x_2)$ which is a function of position. To find the paths which enable it to reach a given curve $\mathscr{C}$ in the least time.

Thus the KE are

$$\dot{x}_1 = w \cos \phi$$
$$\dot{x}_2 = w \sin \phi$$

and $G = 1$.

Problem II. The same, except that the curvature of $\mathbf{x}$'s path is now bounded by $1/R$.

The KE here are

$$\dot{x}_1 = w \cos x_3$$
$$\dot{x}_2 = w \sin x_3$$
$$\dot{x}_3 = w(-\phi/R), \qquad -1 \leqslant \phi \leqslant 1$$

and similarly $G = 1$.

The assertion we wish to make here, special instances of which have been encountered earlier, is

The universal surfaces of Problem II are optimal paths of Problem I.

The proof rests in part on

Exercise 7.12.1. After adjoining $\dot{x}_4 = 1$ to the KE of Problem II, show that the C_{uv} yield

$$w \frac{\breve{\phi}}{R} = w_2 \cos x_3 - w_1 \sin x_3 \qquad (7.12.1)$$

(Here $w_i = \partial w / \partial x_i$).

Another part depends on the less routine

Exercise 7.12.2. For Problem I along an optimal path show that

$$\mathring{\bar{\phi}} = w_2 \cos \bar{\phi} - w_1 \sin \bar{\phi}. \tag{7.12.2}$$

This can be done by our standard procedure, which furnishes first $\bar{\phi}$ as a function of the x_i and V_i and, secondly, supplies $\mathring{x}_i$ and $\mathring{V}_i$ (the RPE) as functions of the same variables, by differentiating the former functions. But other means are possible, for (7.12.2) is tantamount to a second-order differential equation in the x_i alone and is the equivalent of the Euler equation for Problem I.

Now if we replace ϕ by the ϕ of (7.12.1) in the KE of Problem II we have the differential equations of the paths which comprise the US. The third of these, $\mathring{x}_3 = w\breve{\phi}/R$, shows that x_3 satisfies (7.12.2). Thus, through a suitable correspondence of initial conditions, the x_3 of Problem II will be identical with the $\bar{\phi}$ of Problem I.

The interpretation of all this has been foreshadowed. As in Example 7.5.2, the tributaries to the US, where $\bar{\phi} = \pm 1$, are sharpest possible turns. That is, in Problem II, they correspond to an early maneuver to obtain quickly a state—attitude and position—from where **x** can follow an optimal trajectory of Problem I. And these positions constitute the US.

In broader terms, we see that the US of one game may be tantamount to the solutions of another of greater dimension. It would be interesting to explore the generality of this phenomenon, but even now we can perceive why the construction of US constitute a formidable problem. They bear a kinship to integrals of the Euler equation, and we should expect a full theory to be beset by all the complexities of the calculus of variations.

7.13. RESTORATION OF THE TOTALITY OF CONTROL VARIABLES

For reasons already explained, we have confined the treatment of US, of the linear type, to those entailing a single control variable which has been termed ϕ. We now revert to the general situation with the full complement of them. However, one will still play a chief role, still termed ϕ (or ψ if it is maximizing) and our subject will be a ϕ-US.

We will thus adopt here a deviant from our usual notation: the above ϕ will be written without a subscript and $\phi_1, \ldots, \phi_\lambda, \psi_1, \ldots, \psi_\kappa$ will denote the *other* control variables. The linearity of KE in ϕ is still essential; their form is still

$$\dot{x}_i = \alpha_i \phi + \beta_i.$$

We will make an assumption whose strength is between that of the minimax (Section 2.4) and a demand that the KE be separable in all control variables.

The α_i are free of the ϕ_k, ψ_l. (7.13.1)

That is, α_i is at most a function of the x_i, whereas β_i may involve x_i, ϕ_k, ψ_l.

Then the general condition for ϕ-US (or better, a ϕ-CUS) is the following generalization of (7.4.8):

$$\begin{aligned} \sum_i \alpha_i V_i &= 0 \quad (\alpha) \\ \min_{\phi_k} \max_{\psi_l} \sum_i \beta_i V_i &= 0 \quad (\beta) \\ \sum_i \gamma_i V_i &= 0. \quad (\gamma) \end{aligned} \tag{7.13.2}$$

Here (α) is precisely the same as its earlier counterpart. In (β) the min and max are taken in the same sense as has been usual in our theory: they determine each ϕ_k and ψ_l as functions $\bar{\phi}_k$ and $\bar{\phi}_l$ of the x_j and V_i, as in our earlier derivation of the ME; (β) can be written also as

$$\sum_i \beta_i(\bar{\phi}_k, \bar{\psi}_l, x_j)V_i = 0$$

analogously to the ME$_2$. In (γ), γ_i still means

$$\sum_j (\alpha_{ij}\beta_j - \beta_{ij}\alpha_j)$$

but in forming the derivatives, such as $\beta_{ij} = \partial\beta_i/\partial x_j$, any appearing ϕ_k or ψ_l are treated as constants. The latter are *then* replaced by the above $\bar{\phi}_k$ and $\bar{\psi}_l$ functions of the x_j and V_i.

The proof is substantially the same as before. The general ME is

$$\begin{aligned} &\min_{\phi,\phi_j} \max_{\psi_k} \sum_i (\alpha_i\phi + \beta_i)V_i \\ &= \min_{\phi} \phi(\sum \alpha_i V_i) + \min_{\phi_j} \max_{\psi_k} \sum \beta_i V_i \\ &= \min \phi A \qquad + \min \max B = 0. \end{aligned}$$

As in Section 7.4, an interior minimizing ϕ demands that both terms vanish; thus (α) and (β). Finally (γ) follows as earlier by the demand that $\mathring{A}$ be 0 on the CUS. For on the tributary paths we must have our usual RPE, which here are

$$\begin{aligned} \mathring{x}_i &= -(\alpha_i\sigma + \beta_i), \qquad \sigma = -\operatorname{sgn} A \\ \mathring{V}_j &= \sum_i V_i(\alpha_{ij}\sigma + \beta_{ij}). \end{aligned}$$

In the second line the derivatives with respect to x_j are reckoned with the ϕ_l and ψ_k behaving as constants. Thus the stage is set for a repeat performance of the calculation of $\mathring{A}$.

But when we try for a repetition of workable conditions, such as we did earlier for $n = 3$ and 4, we encounter difficulties. Even the former simplicity of the $n = 3$ case is gone, for the equations (7.13.2) are no longer linear in the V_i. But they are still homogeneous, that is, are satisfied when all $V_i = 0$. The new criterion will be the condition on the x_i which ensures that nonzero V_i solutions exist. The ensuing examples will display several possibilities.

Actually many particular problems yield without a panoply of formal ideas. It may be that the restored control variables are few in number and the structure of the solution is apparent. For instance, if these variables are all involved lineally, then each can have but two possible values, and which of these is correct may be inferred from direct simple considerations.

It should be noted that when we derive the ME in our standard manner, (β) will still hold. This is because of our assumption (7.13.1). Thus, as the V_i are continuous across a US, the restored control variables will be also.

Example 7.13.1. A generalization of Examples 7.3.1 and 7.5.1. We return to our former problem with the 45°-triangle vectograms. These belong to the minimizing player P. But now we adjoin a maximizing player E who will have circular vectograms with radius $v(x_1, x_2)$, a smooth function over the plane. The resulting velocity of **x** is to be the sum of the selections from the respective vectograms. The KE then are (compare those of Example 7.5.1)

$$\dot{x}_1 = \phi u + v \sin \psi$$
$$\dot{x}_2 = -u + v \cos \psi$$
$$\dot{x}_3 = 1$$
$$-1 \leqslant \phi \leqslant 1;\ u > 0, v > 0;\ \text{assume always } v < u.$$

Computing the γ_i and tabulating:

i	α_i	β_i	γ_i
1	u	$v \sin \psi$	$u_1(v \sin \psi) + u_2(-u + v \cos \psi) - uv_1 \sin \psi$
2	0	$-u + v \cos \psi$	$-(-u_1 + v_1 \cos \psi)u$
3	0	1	0

Our equation (β) is here

$$\max_{\psi} [v(V_1 \sin \psi + V_2 \cos \psi) - uV_2 + V_3] = 0.$$

As usual, putting $\rho = \sqrt{V_1^2 + V_2^2}$,

$$\sin \bar{\psi} = \frac{V_1}{\rho}, \qquad \cos \bar{\psi} = \frac{V_2}{\rho}$$

and (β) is

$$v\rho - uV_2 + V_3 = 0.$$

But (α) is $uV_1 = 0$ and as $u > 0$, we have $V_1 = 0$. As $v < u$, **x** always moves downward and so $V_2 > 0$. Thus $\rho = V_2$ and $\bar{\psi} = 0$. That is, on a US, E always strives to move **x** vertically upward.

Thus (γ) is

$$V_2(u_1 - v_1)u = 0$$

and the sought US condition is

$$u_1 - v_1 = 0.$$

This is a "necessary" condition only, and remarks made earlier on this point apply here also.

Example 7.13.2. The homicidal chauffeur game. We return to Example 7.5.2. which deals with the car without the pedestrian and adjoin the latter. As above, this means that we add to the KE terms corresponding to a circular vectogram of radius w_2, the speed of the evader, and we write once more the KE of the title game.

$$\dot{x}_1 = -\frac{w_1}{R} x_2\phi + w_2 \sin \psi,$$
$$\dot{x}_2 = \frac{w_1}{R} x_1\phi - w_1 + w_2 \cos \psi$$
$$\dot{x}_3 = 1, \qquad -1 \leqslant \phi \leqslant 1,\ w_1 > w_2 > 0.$$

The next step is to form the matrix:

i	α_i	β_i	γ_i[14]
1	$-(w_1/R)x_2$	$w_2 \sin \psi$	$-(-w_1 + w_2 \cos \psi)$
2	$(w_1/R)x_1$	$-w_1 + w_2 \cos \psi$	$w_2 \sin \psi$
3	0	1	0.

From (β) we obtain, just as in the previous example,

$$\sin \bar{\psi} = \frac{V_1}{\rho}, \qquad \cos \bar{\psi} = \frac{V_2}{\rho}. \tag{7.13.3}$$

[14] We suppress the common factor w_1/R_1.

Then (γ) yields

$$V_1\left(w_1 - w_2\frac{V_2}{\rho}\right) + V_2\left(w_2\frac{V_1}{\rho}\right) = V_1 w_1 = 0$$

so that $V_1 = 0$. If also $V_2 = 0$, (β) would imply the then absurd consequence $V_3 = 0$; thus $V_2 \neq 0$. As (α) is

$$x_1 V_2 = 0$$

we have as a CUS:

$$x_1 = 0.$$

This is the same result that we found in Example 7.5.2 and the discussion there largely applies. Optimal play will be of the direct type already depicted in Figure 1.5.2a.[15]

To see that this is so we must study the tributary paths to the US, but we do not require a full analysis. We are mainly interested in ψ, E's optimal travel direction.

Recalling that the ME$_2$ is (we return to the x, y symbols)

$$-\frac{w_1}{R}[yV_x - xV_y]\bar{\phi} - w_1 V_y + w_2\rho + 1^{16} = 0$$

and, as on the US, $A = 0$, the bracket vanishes. As $x = 0$, $y > 0$ on the US, there we have $V_x = 0$. From our standard procedure

$$\sin\bar{\psi} = \frac{V_x}{\rho} = 0, \qquad \cos\bar{\psi} = \frac{V_y}{\rho} > 0 \tag{7.13.4}$$

we have $\bar{\psi} = 0$ (clearly $V_y > 0$ on the US). Consequently, the points of the US correspond to P's traveling straight and E also, fleeing directly from P.

Turning to the tributaries, we find among the RPE

$$\overset{\circ}{V}_x = \sigma\frac{w_1}{R}V_y, \qquad \overset{\circ}{V}_y = -\sigma\frac{w_1}{R}V_x$$

where $\sigma = \pm 1$ depending on the side of the US. These equations imply that the vector V_x, V_y rotates with angular speed w_1/R. But E's velocity, as seen from (7.13.4), lies along this vector and it too so rotates. But such is also the rotation of P on his sharp turns, so that E's velocity does not turn relative to P; in the realistic space E's path is straight.

Finally, as V_x and V_y are continuous at the US, so is E's flight direction. It follows that E's original velocity must lie on the tangent to one of the minimal curvature circles. Thus the direct play, which we discussed in Section 1.5, is optimal.

[15] In Chapter 10 we shall learn that part of the *lower* x_2-axis can also be universal. But in an inverted sense, as E now pursues P.

[16] In the terminal payoff form, this 1 is replaced by V_3.

Research Problem 7.13.1. The asymmetric homicidal chauffeur game. Let w_2, the evader's speed, be, instead of constant, a function u of x_1 and x_2. Show that the condition for a US is now

$$w_1 x_1 + (u_1 x_2 - u_2 x_1)\sqrt{x_1^2 + x_2^2} = 0^{17} \tag{7.13.5}$$

The speed of E has been made a function of his coordinates relative to P. A fanciful way of realizing this problem is to think of P as a car with headlights (whose beams are perhaps asymmetrical) and the chase taking place on rough terrain in the dark. The evader's speed is the higher the more he is illuminated. Thus P has something to gain by pointing askew and keeping his lights off his quarry.

Since the equations (7.13.2) are homogeneous in the V_i, when $n = 3$, (α), being of degree 1, can be used to eliminate one of the V_i, say V_3, from (β) and (γ). These two can then each be written as an equation in V_1/V_2. Our workable condition will then be that these equations possess a common real root.

Example 7.13.3. Let the KE be

$$\dot{x}_1 = x_3^2\phi_1 + \cos\phi_2 + x_2\psi$$
$$\dot{x}_2 = \phi_1 + x_2$$
$$\dot{x}_3 = x_1\sin\phi_2.$$

Trying for a ϕ_1-US leads to

$$x_3^2 V_1 + V_2 = 0 \quad (\alpha)$$
$$x_2 V_1\sigma + x_2 V_2 - \rho = 0 \quad (\beta)$$
$$-\left(\frac{x_1 x_3 V_3}{\rho}\right)(2x_1 V_1 - x_3 V_3) - (\sigma V_1 + V_2) = 0 \quad (\gamma)$$

where $\rho = \sqrt{V_1^2 + (x_1 V_3)^2}$, $\sigma = \text{sgn}(x_2 V_1)$.

Eliminating ρ between (β) and (γ) gives us an algebraic equation in the V_i. Removing the ρ from (β) by squaring gives us another. We end up with three homogeneous algebraic equations: (α) and

$$x_1 x_3 V_3(2x_1 V_1 - x_3 V_3) + x_2(\sigma V_1 + V_2)^2 = 0$$
$$x_2^2(V_1\sigma + V_2)^2 = V_1^2 + x_1^2 V_3^2.$$

Eliminating the V_i as above leads to

$$x_1[x_2 x_3(2x_1^2\sigma_1\sqrt{C^2 - 1} - x_3(C^2 - 1)) + C^2 x_1] = 0$$

[17] Assuming $x_2 v_2 > 0$; otherwise the + is changed.

where $C = x_2(\sigma_1 - x_3^2)$, $\sigma_1 = \text{sgn}(x_1 V_3 C)$, an algebraic equation of a CUS.

Exercise 7.13.1. Show that, in the previous example, the condition for a ψ-US is similarly homogeneous and find the ultimate equation.

Research Problem 7.13.2. Is it always true that (7.13.2) are homogeneous in the V_i?

7.14. SEMIUNIVERSAL SURFACES

Such are singular surfaces of type $(+, u, p)$, that is, tributary paths appear on one side only; parallel ones on the other. We display their existence by the following example.

Example 7.14.1. The gunner and approaching target.[18] The KE are

$$\dot{x} = -1$$

$$\dot{m} = -\psi, (0 \leqslant \psi \leqslant 1).$$

The payoff is integral with $G = \psi g(x)$; $\mathscr{E}$ is the first quadrant: $x \geqslant 0$, $m \geqslant 0$ while $\mathscr{C}$ is its boundary, the two positive half-axes. Further, $g(x)$ is positive, smooth, and decreasing.

The interpretation is of a gunner located at the origin ($x = 0$), firing at an approaching target at x, this coordinate diminishing at a unit rate. At any time the gunner has a supply m of ammunition which he is at liberty to fire at any rate from zero to unity. The control variable ψ is this rate; m is a state variable.

It can be shown reasonably[19] that the kill probability is an increasing function of $\int \psi g(x)\,dx$, the integral extending over the partie, for a suitable function g as above. Thus the above G.

The optimal strategy could hardly be more obvious: the gunner waits until the target is as near as the ammunition allows, that is, until he can just sustain full fire throughout the final interval of the target's approach. The optimal paths, then, are as in Figure 7.14.1.[20]

Our interest is the semiuniversal surface, the line OA in the figure. Tributary paths appear on one side only.

It is clear that if $\mathscr{C}$ were suitably modified, any of the 45° paths could be made to play the semiuniversal role. Note the contrast to US, where only a small number of rather special surfaces are usually eligible and they are

[18] A simplification of one case of the bomber and battery game in the Appendix, Example A5.3.

[19] See Appendix A1.

[20] Compare Figure A5.4.

determined from the KE and are in good part independent of $\mathscr{C}$ or $\mathscr{E}$. This distinction precludes a clean, sharp theory for semiuniversal surfaces.

We work, on the same grounds as for US, with a game of terminal payoff with $n \geqslant 3$, one control variable ϕ, and linear vectograms. That is, as before, the KE are

$$\dot{x}_i = \alpha_i \phi + \beta_i, \qquad -1 \leqslant \phi \leqslant 1$$

For a possible semiuniversal surface we select a region $\mathscr{R}$ on a smooth surface $\mathscr{S}$ comprised of optimal paths emanating from an initial surface $\mathscr{C}$. On $\mathscr{R}$, we suppose regular behavior; say $\bar{\phi} = -\sigma = -\text{sgn}\, A$. We wish to know conditions permitting the use of $\mathscr{R}$ as an initial surface for optimal paths with $\bar{\phi} = +\sigma$. For quantities pertinent to these new paths we shall use notations V^*, A^*, etc.

THEOREM 7.14.1. Sufficient conditions for $\mathscr{R}$, as above, to be a semiuniversal surface are

(1) Throughout $\mathscr{R}$, the plane of the vectogram is not tangent to $\mathscr{S}$.

(2) On $\mathscr{R}$, $\text{sgn} \sum_i \gamma_i V_i^* = \sigma$. (The V_i^* can then be intrinsically determined on $\mathscr{R}$ as shown below.)

Proof. Let us select the parameters $s_1, \ldots, s_{n-1}$ of $\mathscr{C}$ so that for $\mathscr{S} \cap \mathscr{C}$, $s_{n-1} = 0$. On this curve let H be $J(s_1, \ldots, s_{n-2})$. The parametric equations of $\mathscr{R}$ are then

$$x_i = x_i(\tau, s_1, \ldots, s_{n-2}) \tag{7.14.1}$$

where the right sides are integrals of the RPE with the usual initial conditions, that is, as determined from H on $\mathscr{C}$. These RPE also imply that

$$\overset{\circ}{x}_i = -\sigma\alpha_i + \beta_i.$$

On $\mathscr{S}$,

$$V = J(s_1, \ldots, s_{n-2}). \tag{7.14.2}$$

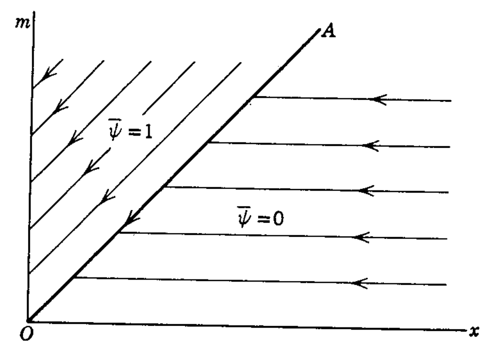


Figure 7.14.1

We are going to use (7.14.1) and (7.14.2) as initial conditions in the usual way, but for the paths obtained we must have $\bar{\phi} = +\sigma$. The customary initial conditions, relevant to the parameters $s_1, \ldots, s_{n-2}, \tau$ are then

$$\frac{\partial V}{\partial s_k} = \frac{\partial J}{\partial s_k} = \sum_i x_{ik} V_i^*, \qquad x_{ik} = \frac{\partial x_i}{\partial s_k}, \quad k = 1, \ldots, n-2$$

$$\frac{\partial V}{\partial \tau} = 0 = \sum_i (-\sigma\alpha_i + \beta_i) V_i^*$$

and we adjoin the ME for the new system

$$0 = \sum_i (\sigma\alpha_i + \beta_i) V_i^*.$$

The last equations are equivalent to

$$\sum_i \alpha_i V_i^* (= A^*) = \sum_i \beta_i V_i^* = 0,$$

the determinant

$$\begin{vmatrix} x_{i1} \\ \cdot \\ \cdot \\ \cdot \\ x_{i,\, n-2} \\ \alpha_i \\ \beta_i \end{vmatrix} \neq 0$$

in virtue of (1). Thus we can find the V_i^*. As $A^* = 0$, $\bar{\phi} = -\operatorname{sgn} \mathring{A}^*$ if the latter $\neq 0$. But (2) requires that

$$\sigma = -\operatorname{sgn} \mathring{A}^* = \operatorname{sgn} \sum_i \gamma_i V_i^*$$

and this suffices for at least a local path construction.

COROLLARY 7.14.1. On a semiuniversal surface,

$$A^* = B^* = 0.$$

Suppose $\mathscr{R}$ is not all of $\mathscr{S}$ through failure of the condition (2). Then on the boundary of $\mathscr{R}$, we should expect that

$$\sum_i \gamma_i V_i^* = 0. \tag{7.14.3}$$

But this equation and the two of the corollary are just our necessary conditions for a US, implying that one may emanate from the boundary of $\mathscr{R}$. For a typical instance of such a possibility, see Figure 7.14.2.

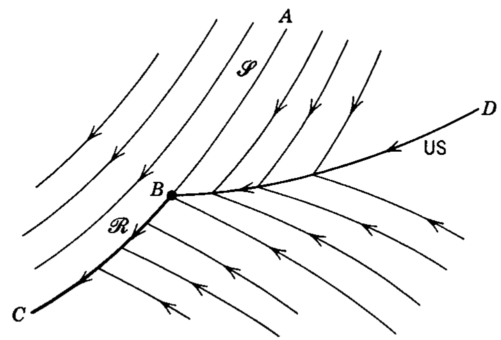


Figure 7.14.2

Here $\mathscr{S}$ is AC, but $\mathscr{R}$ is BC where (2) holds. At B, then, we have (7.14.3). The US is curve BD. Its tributaries on the lower side merge with those of $\mathscr{R}$; those above the US are an extension of family of paths on the far side of $\mathscr{S}$.

Observe that if (1) and (2) hold on a surface, they hold on neighboring surfaces. Thus candidates for semiuniversal surfaces occur in families.

It is hard to say much more. Of the abundance of candidates, the few, if any, will depend on $\mathscr{C}$ and $\mathscr{E}$. As in our example, such a one must be the extreme of a family bounding a void (or "semivoid").

The wonder is, that considering their profilic possibilities, semiuniversal surfaces have appeared much more rarely than US in the examples we have considered, both in and out of this book.

An interesting possibility appears in Example A4 of the Appendix. There $\mathscr{C}$ is at first a circle and two transition curves emanate from it. As the radius becomes zero, each curve becomes semiuniversal.

Problem 7.14.1. Taking the KE of Example 7.14.1

$$\dot{x}_1 = -1$$
$$\dot{x}_2 = -\psi$$
$$\dot{x}_3 = \psi g(x_1)$$

let $\mathscr{C}$ be $x_1 = 0$; $\mathscr{E}$, $x_1 \geqslant 0$. Take $H = x_1$ and suppose that $g(\tau) > 0$ for $0 \leqslant \tau < \tau_0$ but $g(\tau_0) = 0$. Take $\mathscr{S}$ as a family of optimal paths through the curve in $\mathscr{C}$: $x_1 = 0, x_2 = s, x_3 = f(s)$ where $f'(s) > 0$. Show that $\mathscr{S}$ is a possible semiuniversal surface as long as $\tau < \tau_0$.

CHAPTER 8

Games of Kind

Typical of a game of kind, in distinction to one of degree, is a pursuit game in which we are interested in what conditions make capture possible for the pursuer or escape for the evader, rather than seeking the best procedures in terms of optimizing some continuous payoff.

Until now there was, as far as we know, only one approach to such problems. We have dubbed it the *Method of the Explicit Policy*. It consists of, say, demonstrating the possibility of capture by exhibiting a particular strategy or policy of the pursuer which attains it. Its weakness is that, in almost all positions, there is no determinate best decision for either player; to seek a sequence is to grope in the dark. One can seldom solve all cases of one particular problem, let alone solve all problems of a class.

The innovation here emphasizes the hypersurface, called the barrier, which separates, in the space of starting positions, those of capture from those of escape. For starting points *on* the barrier, optimal behavior leads to a contact of the terminal surface without a penetration. We term such an outcome *neutral* and regard it as intermediate between capture and escape. The advantage is that for neutral outcomes alone do there exist determinate optimal strategies.

Differential equation techniques, similar to those of games of degree, lead to optimal strategies and paths and thence to the barrier. The global answer to capture-or-escape question then hinges on whether or not the barrier divides the playing space into two parts.

One of the chief difficulties is the finding of the proper initial conditions or the means of attaching the barrier to the terminal surface. We have

discovered three, all rather different, one in particular being tantalizingly subtle. Between them they cover all practical instances thus far encountered but do not appear to be exhaustive.

Significant examples of problems solved by these methods will appear in the following chapter. Space here has limited us to simple illustrative instances.

8.1. INTRODUCTION

In games of degree, we recall, the players strive to maximize and minimize a certain payoff capable of assuming a continuum of numerical values. For any partie, the particular value of the payoff is not ascertained until termination, that is when $\mathbf{x}$ reaches the terminal surface $\mathscr{C}$. Thus our whole theory of such games is erected on the presumption that $\mathscr{C}$ is actually reached.

In this chapter we discuss games of kind, in which the achievement of termination itself is the quintessence of the problem.

We shall almost always suppose that one player wishes termination and his opponent does not: these opposing desiderata are the conflict of the game. But other circumstances are possible and indeed useful. For example, collision avoidance between two moving craft.[1] Both pilots wish to prevent "termination" (= collision in our format). Although such a situation is not a differential game, it is amenable to the same techniques.

We shall use the vernacular of pursuit games as a surrogate for all cases. That is, we will assume that it is P who desires termination—capture in a pursuit game—and E, its avoidance.

To subsume these games technically into our general scheme we can assign numerical values to the outcomes, retaining the concept of the payoff. For example,

$+1$ for no termination or *Escape*

-1 for termination or *Capture*

enable P and E to remain the minimizing and maximizing players.

With such payoffs, the general theory of games defines the Value as

$+1$ if there exist a strategy for E such that, when he plays it, termination will never occur no matter what the strategy of P. (8.1.1)

-1 if there exist a strategy for P such that, if he plays it, termination is certain to occur no matter what the strategy of E. (8.1.2)

[1] To appear at a later date.

Although we shall not require the formalism of numerical payoffs in practice, note that it is consistent with the general nomenclature to define the two particular strategies mentioned as *optimal strategies* for E and P.

In contrast to games of degree, in most cogent examples the optimal strategies are not unique. Indeed they are legion. Consider, for example, any reasonably simple pursuit game in which P is so kinematically superior to E that he can capture from any starting position ($V(\mathbf{x}) = -1$ for all $\mathbf{x} \in \mathscr{E}$). As we are interested only in capture at *some time*, P may loiter as long as he likes; there are never preferred values of the control variables. Similarly, if the superiority attaches to E, so that escape is always possible, E has complete freedom of action save possibly when capture is imminent and even then he may evade with as close a shave as he pleases.

To prove that P can capture or E escape, it suffices to exhibit one particular strategy that will enable him to do so despite any opposition. This most obvious technique for games of kind we shall call the *Method of the Explicit Policy*.[2] An instance is:

Example 8.1.1. The game of two cars. The points P and E move in a plane each at a fixed (or bounded) speed and each with its curvature bounded. The four bounds may be any magnitudes. Capture, as usual, means $|PE| \leqslant l$. Under what conditions can P capture E?

If P has the higher speed and at least as favorable a curvature restriction as E, capture can be attained. For P can first go to E's starting position and then follow his track.

Generally such a display of a particular policy suffers the drawback of there being no systematic way to find one. We have to be ingenious anew in each case. For example, it seems reasonable that P could capture with a slight inferiority of curvature provided he was sufficiently superior in speed and l is generously large. How do we exhibit a policy to prove it?

We offer in this chapter what appears to be an approach of greater generality, with wide, if not universal, applicability. By a slight but natural alteration of the criterion, we incur a subset of $\mathscr{E}$ on which optimal strategies are determinate if not unique. Artificial, inefficient policies, as that of P above, are avoided.

8.2. THE BARRIER CONCEPT

First we modify, for a definite purpose, the definition of termination. We demand that $\mathbf{x}$ not only reach $\mathscr{C}$, but also penetrate it. Should $\mathbf{x}$ reach $\mathscr{C}$ without crossing it and ultimately return into $\mathscr{E}$, the outcome will be considered as neither capture nor escape but as a third alternative. Numerically we can say that now

$$\text{payoff} = 0.$$

We will refer to this third possible outcome as *neutral*. It is to be regarded as the delineating case between capture and escape. Our motive is that only in cases of neutrality ($V = 0$) is each player's actions, as he is on the verge of a worse payoff throughout play, decisive. Generally it is in this case alone that there exist definite and determined optimal strategies. Here there is material for a calculable theory.

We recall that a differential game is really a family of games, one for each starting position. One of three possibilities must occur:

(E) (8.1.1) holds for all starting points of $\mathscr{E}$.
(C) (8.1.2) holds for all starting points of $\mathscr{E}$.
(M) $\mathscr{E}$ contains both kinds of starting points.

The set of starting points for which (8.1.1) holds we will call the escape zone (EZ); those for which (8.1.2), the capture zone (CZ).

When (M) prevails, the two zones will each be a region. Generally they will be separated by a surface which consists of the starting points for which the outcome is neutral. This surface will be called the *barrier*.

The nucleus of our approach will be the ascertainment of the barrier. Knowledge of it will automatically engender that of the CZ and EZ and so distinguish the situations from which P can force capture or E escape whenever (M) obtains. And the latter happens in the most interesting cases, for when (E) or (C) holds, the situation is often one sided enough to be transparent.

Even when (M) does not obtain, our ideas can apply. For the game will always entail parameters—speeds, capture radius, etc.—which appear formally in the coefficients in the KE, H or G or specification of $\mathscr{C}$. By varying them we can often occasion (M) when it did not pertain originally. Thus we imbed the game in a continuum by varying some or all of the parameters.

For example, we have seen an instance of the game of two cars in which (C) holds (the capture zone $= \mathscr{E}$). Now suppose E has the greater speed. If initially E is pointed away from P, he can escape by going straight. But for a starting position in which P and E point toward one another and are sufficiently close, it is intuitively clear that capture ensues. Thus both zones are not vacuous. There will be a surface $\mathscr{B}$, the barrier, separating them.

Now starting from such a case, let us continuously vary the parameters

[2] As the ensuing example will show, it may be convenient to admit playing policies which are not strategies.

in some way that tends to favor P. We know that this can be done so that at some point $\mathscr{E}$ becomes all capture zone. What happens to the barrier then, which we suppose has deformed continuously? It will, at some critical point, have ceased dividing $\mathscr{E}$ into two parts.

This situation is general. We shall extend the term barrier to include surfaces which do not divide $\mathscr{E}$ into two parts, when such a surface derives from the barrier (in the original sense) through a continuous change in the parameters. This definition may appear vague; what we have in mind is the result of a construction which will be given shortly.

The barrier in this extended sense—when it does not necessarily delineate the capture and escape zones—still constitutes, as we shall see, an important singular surface when certain basic continuous payoffs are adopted and the game becomes one of degree. Generally the barrier is then of type $(p, -)$ or $(-, -)$, is not crossed during optimal play, and marks a discontinuity in V. Although it may not bound an escape zone, it does delineate a region in which P's task is more difficult.

Inasmuch as there will be determinate best ϕ and ψ only when the outcome is neutral, that is, $\mathbf{x}$ is on the barrier, we shall use the term optimal strategy when dealing with games of kind in this limited way. That is, $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ are now defined only for $\mathbf{x}$ of the barrier.

Perhaps some will find the distinction of the neutral outcome objectionable. As $\mathscr{C}$ is a closed set of $\mathscr{E}$, any meeting of $\mathbf{x}$ with $\mathscr{C}$ should be capture. Penetration of $\mathscr{C}$ is meaningless in that the no KE are postulated for $\mathbf{x}$ outside of $\mathscr{E}$.

These objections can be assuaged by the following modification of concept which permits retention of the convenient barrier methodology. Let us think of a surface $\mathscr{C}_\varepsilon$ parallel to $\mathscr{C}$, a distance ε away, and within $\mathscr{E}$. We shall assume that the continuity of the KE causes all barrier constructions based on $\mathscr{C}_\varepsilon$, to be near those of $\mathscr{C}$ if ε is small. Let E, starting from some $\mathbf{x}$ of the escape zone, pick ε so small that $\mathbf{x}$ is in the escape zone relative to $\mathscr{C}_\varepsilon$; then let him play as if $\mathscr{C}_\varepsilon$ were the terminal surface. His strategy will be arbitrary unless $\mathbf{x}$ is on the new barrier; there he plays optimally. But a neutral outcome in the new game means escape in the old. If $\mathbf{x}$ is in the capture zone, P plays optimally as in the original game.

Very basic is the fact that

The barrier is a semipermeable surface.

For let $\mathbf{x}$ be on the barrier separating the escape and capture zones. Now P must be able to select a value $\bar{\phi}$ of ϕ which prevents $\mathbf{x}$ from entering the escape zone, for otherwise E could attain escape starting from $\mathbf{x}$. Similarly, E can keep $\mathbf{x}$ out of the capture zone. These abilities to prevent penetration comprise the definition of a semipermeable surface.

Note that, in addition, a particular orientation of this surface is required; namely, the penetration direction which E can prevent must lead into the capture zone.

Thus we must next turn to

8.3. THE CONSTRUCTION OF SEMIPERMEABLE SURFACES

Let $\mathscr{S}$ be a smooth surface in $\mathscr{E}$, and at each of its points let $\nu = (\nu_1, \ldots, \nu_n)$ be its normal vector. That is, each ν_i is a function of the x_i defined when $\mathbf{x} \in \mathscr{S}$. The length of ν, as long as it is not zero, is arbitrary; it can be taken as any convenient nonzero function of $\mathbf{x}$ when working problems. But the orientation of ν is important. Thus ν is determined up to multiplication by an arbitrary positive function of $\mathbf{x}$.

The condition that $\mathscr{S}$ be a semipermeable surface

$$\min_{\phi} \max_{\psi} \sum_{i=1}^{n} \nu_i f_i(\mathbf{x}, \phi, \psi) = 0. \tag{8.3.1}$$

This equation can be regarded as our usual ME with a change of interpretation. Indeed (8.3.1) will be referred to as the main equation. However, we shall give an independent derivation.

As f_i is $\dot{x}_i$, the ith component of the velocity of $\mathbf{x}$ when ϕ and ψ are played, the sum in (8.3.1) is the component of this velocity in the direction of ν. Then this sum's being >0 [<0] is equivalent to $\mathbf{x}$'s penetration of $\mathscr{S}$ in the [opposite] direction of ν. Now let $\bar{\phi}$ supply the min in (8.3.1); we have

$$0 = \max_{\psi} \sum_{i} \nu_i f_i(\mathbf{x}, \bar{\phi}, \psi) \geqslant \sum_{i} \nu_i f_i(\mathbf{x}, \bar{\phi}, \psi)$$

for any ψ on the extreme right. Thus the use of $\bar{\phi}$ assures no penetration in the ν direction, for any ψ. Putting the same shoe on the ψ foot concludes the proof.

We select a definite $\bar{\phi}$ and $\bar{\psi}$ for each point of $\mathscr{S}$ which furnishes the min and max in (8.3.1). Furthermore, we suppose them reasonably smooth functions over $\mathscr{S}$. Then at each point of $\mathscr{S}$, $\mathbf{x}$ will have a definite velocity $\{f_i(\mathbf{x}, \bar{\phi}, \bar{\psi})\}$.

It may be that this velocity is everywhere zero on $\mathscr{S}$. In this case we shall call the semipermeable surface *static*. The equivalent formal condition is obviously

$$f_i(x, \bar{\phi}, \bar{\psi}) = 0, \qquad i = 1, \ldots, n. \tag{8.3.2}$$

Example 8.3.1. Let $n > 1$ and P and E each choose unrestrictedly the direction of a unit velocity vector, the net velocity of $\mathbf{x}$ being the vector sum of these two choices. Then, obviously, any smooth surface is semipermeable. The players choose velocities lying along ν but in opposite directions. It follows that the surface is static.

But the case where (8.3.2) nowhere holds is much the more interesting. Assuming a nonzero velocity at each point of $\mathscr{S}$, from (8.3.1) it is clear that this velocity will be tangent to $\mathscr{S}$ there. From the existence theorem for differential equations, we see that $\mathscr{S}$ will thus be a union of paths such as are described by $\mathbf{x}$ when governed by the strategies $\bar{\phi}$ and $\bar{\psi}$.

On the assumption that $\mathscr{S}$ can be imbedded in a family of semipermeable surfaces which univalently fill a neighborhood $\mathscr{N}$ of it, we can derive a set of RPE for these paths. The derivation contains no formal novelty. It is the same as was done before with the ν_i replacing the V_i.

Explicitly, the required assumption is that ν can be extended throughout $\mathscr{N}$. Choose particular smooth $\bar{\phi}(\mathbf{x}, \nu)$, $\bar{\psi}(\mathbf{x}, \nu)$ satisfying (8.3.1) and defined for $\mathbf{x} \in \mathscr{N}$. Write (8.3.1) in the "ME$_2$ form"

$$\sum_i \nu_i f_i(\mathbf{x}, \bar{\phi}, \bar{\psi}) = 0. \tag{8.3.3}$$

Then let ν_i be functions on $\mathscr{N}$ satisfying (8.3.3) there and equaling the original ν_i on $\mathscr{S}$. Such is the needed extension.

Let $\bar{\phi}$ and $\bar{\psi}$, functions of x_i, ν_i, furnish the min and max in (8.3.1). We differentiate $\sum_i \nu_i f_i(\mathbf{x}, \bar{\phi}, \bar{\psi})$ with respect to x_j and consider the different type terms. First, those due to the externally appearing ν_i give, where ν_{ij} means $\partial \nu_i / \partial x_j$:

$$\sum_i \nu_{ij} f_i(\mathbf{x}, \bar{\phi}, \bar{\psi}). \tag{8.3.4}$$

The terms due to the explicitly appearing x_j in the f_i are

$$\sum_i \nu_i f_{ij}(\mathbf{x}, \bar{\phi}, \bar{\psi}). \tag{8.3.5}$$

Next, those x_j appearing as arguments in $\bar{\phi}(x_j, \nu_i(x_j))$ which are

$$\sum_i \nu_i \sum_k \frac{\partial f_i}{\partial \phi_k} \frac{\partial \bar{\phi}_k}{\partial x_j}$$

which can also be written as

$$\sum_k \left(\frac{\partial}{\partial \phi_k} \sum_i \nu_i f_i \right) \left(\frac{\partial \bar{\phi}_k}{\partial x_j} \right).$$

But (as we reasoned once before), if $\bar{\phi}_k$ is an interior minimum, the first parenthesis vanishes. If it is an exterior one, $\bar{\phi}_k$ remains for all local $\mathbf{x}$ at one of its extreme limits, which we may suppose to be constants. Then the second parenthesis is zero.

Similarly the terms entailing the arguments of $\bar{\psi}$ vanish in any case. Classical methods of analysis can be used to show the existence of $F(\mathbf{x})$ defined on $\mathscr{N}$ such that, if the ν_i are of proper length,

$$\nu_i = \frac{\partial F}{\partial x_i}.$$

This implies

$$\frac{\partial \nu_i}{\partial x_j} = \frac{\partial \nu_j}{\partial x_i}$$

or $\nu_{ij} = \nu_{ji}$.

We make this replacement in (8.3.4), which becomes

$$\sum_i \nu_{ji} f_i$$

which is also

$$\sum_i \frac{\partial \nu_j}{\partial x_i} \frac{dx_i}{dt}$$

the latter derivative pertaining to the motion of $\mathbf{x}$ when $\bar{\phi}(\mathbf{x}, \nu(\mathbf{x}))$ and $\bar{\psi}$ are employed as strategies. But the last form of (8.3.4) can be written as $\dot{\nu}_j$. Thus we are led to

$$\dot{\nu}_j = -\sum_i \nu_i f_{ij}.$$

Finally, we take this equation and the original KE with ϕ and ψ replaced by $\bar{\phi}$ and $\bar{\psi}$ and reverse the time direction (τ replaces t) in both:

$$\mathring{x}_j = -f_j(\mathbf{x}, \bar{\phi}, \bar{\psi}), \qquad \mathring{\nu}_j = \sum_i \nu_i f_{ij}(\mathbf{x}, \bar{\phi}, \bar{\psi}) \tag{8.3.6}$$

our new RPE.[3]

Research Problem 8.3.1. Can the RPE (8.3.6) be derived without utilizing the neighborhood $\mathscr{N}$ of $\mathscr{S}$?

Although this problem is a natural one at this point, its practical consequence is not great for us. To apply our ideas to games of kind, we shall find the membership of $\mathscr{S}$ in a family useful.

In general,[4] we can pass a unique semipermeable surface (with a proper orientation) through a given curve (= $(n-2)$-dimensional manifold in $\mathscr{E}$). Let the latter be

$$\mathscr{D}: \ x_i = h_i(s_1, \ldots, s_{n-2}). \tag{8.3.7}$$

[3] We shall always use the numerical subscript notation for the ν_i.

[4] The scope of the "in general" is like that in the familiar theory of first-order partial differential equations. Any curve will serve as a proper seat of initial conditions if it is nowhere tangent to a characteristic.

We must first fit the ν_i to $\mathscr{D}$. Normality requires that

$$\sum_i \nu_i h_{ij} = 0 \quad \left(h_{ij} = \frac{\partial h_i}{\partial s_j}\right), \quad j = 1, \ldots, n-2 \tag{8.3.8}$$

and the ME_2 (8.3.3) is also to be satisfied on $\mathscr{D}$.

Thus we have $n - 1$ equations to determine the ν_i. One must note whether the orientation of ν fits the problem at hand, but otherwise the length of ν is arbitrary.

The following result is a sort of converse, showing that our construction leads to a semipermeable surface.

THEOREM 8.3.1. Let $\bar{\phi}$ and $\bar{\psi}$ denote functions of the x_i and ν_i which furnish the min and max in (8.3.1). For a curve $\mathscr{D}$ given by (8.3.7) let $\bar{\nu}_i$ be values of ν_i not all zero satisfying (8.3.8) and (8.3.3). Let $x_i(\tau, s_1, \ldots, s_{n-2})$ and $\nu_i(\tau, s_1, \ldots, s_{n-2})$ be integrals of the differential equations (8.3.6) with h_i and $\bar{\nu}_i$ as initial conditions. Then $x_i(\tau, s_1, \ldots s_{n-2})$ is the parametric representation of a semipermeable surface which contains $\mathscr{D}$.

Proof. Let $Q = \sum_i \nu_i f_i(\mathbf{x}, \bar{\phi}, \bar{\psi})$, and let us calculate $\mathring{Q}$, the (retrograde) rate of change of Q along each path [as expressed by the $x_i(\tau)$]:

$$\mathring{Q} = \sum_i \mathring{f}_i \nu_i + \sum_j f_j \mathring{\nu}_j.$$

Now
$$\mathring{f}_i = \sum_j (f_{ij} + \ldots)\mathring{x}_j = -\sum_j f_{ij} f_j.$$

The dots stand for terms such as

$$\frac{\partial f_i}{\partial \phi_1}\frac{\partial \bar{\phi}_1}{\partial x_j}$$

which, as discussed earlier, are all zero. The second equality results from the first of (8.3.6). Thus the first sum of $\mathring{Q}$ is

$$-\sum_{i,j} f_{ij}\nu_i f_j.$$

This is annulled by the second sum, if we replace $\mathring{\nu}_j$ by its value from (8.3.6). Thus $\mathring{Q}$ equals zero. Because Q equals zero on $\mathscr{D}$, it equals zero on $\mathscr{S}$ and (8.3.1) is satisfied there.

Similarly, if

$$R_k = \sum_j \nu_j \frac{\partial x_j}{\partial s_k}$$

then

$$\mathring{R}_k = -\sum_j \nu_j \frac{\partial f_j}{\partial s_k} + \sum_j \left(\sum_i \nu_i f_{ij}\right)\frac{\partial x_j}{\partial s_k}$$

and
$$\frac{\partial f_j(\mathbf{x}, \bar{\phi}, \bar{\psi})}{\partial s_k} = \sum_i f_{ji}\frac{\partial x_i}{\partial s_k} + \text{terms that} = 0 \text{ as before.}$$

This yields $\mathring{R}_k$ equals zero after substitution and an interchange of i, j in one of the two sums. Because R_k equals zero on $\mathscr{D}$, due to (8.3.8), R_k equals zero on $\mathscr{S}$.

These $n - 1$ equations and (8.3.3) holding on $\mathscr{S}$ imply that ν is normal to $\mathscr{S}$. Then (8.3.1) implies that $\mathscr{S}$ is semipermeable. Obviously the points of $\mathscr{S}$ with $\tau = 0$ form $\mathscr{D}$.

The proof also shows that

If $\bar{\phi}$ and $\bar{\psi}$ as determined from (8.3.1) are unique, there is a unique solution for every essentially distinct ν_i satisfying (8.3.8) and (8.3.3).

The semipermeable surface obtained by the foregoing procedure is imbeddable in a neighboring family of such surfaces. It is only necessary to imbed the curve $\mathscr{D}$ in a one-parameter family of nearby curves and use each. The family will constitute a surface and it is, of course, necessary that this surface is not tangent to $\mathscr{S}$.

We have been assuming that $\mathscr{S}$ is smooth. The scope of our work can be broadened to include various types of exceptions. The ideas are as in our previous pages. For let $\mathscr{S}$ be imbedded in a family of semipermeable surfaces. Let us select a smooth function F which is constant on each but with a nonzero rate of change throughout the family. Clearly what we now have is tantamount to the surfaces of constant Value ($V = F$) of a terminal payoff game. To it we may apply our earlier concepts of singular—transition, universal, etc—surfaces. These are reflected as corresponding singular curves on $\mathscr{S}$.

Exercise 8.3.1. Given the KE

$$\dot{x}_1 = \psi^2 - 4x_2$$
$$\dot{x}_2 = -2\psi$$
$$\dot{x}_3 = 1 - \phi - x_1, \qquad -1 \leqslant \phi \leqslant 1^{5}$$

pass a semipermeable surface through the x_2-axis (parameterized $x_1 = 0$, $x_2 = s$, $x_3 = 0$). Two are possible of opposite orientation; settle the choice by assuming $\nu_3 > 0$ (one may take $\nu_3 = 1$). We should have $\nu_1 < 0$ except on $\mathscr{D}$; why?

[5] As long as $\nu_1 < 0$, it is not necessary to place bounds on ψ, as will appear from the calculations.

[The solution is

$$x_1 = 4s\tau - 4\tau^3$$
$$x_2 = s - 2\tau^2$$
$$x_3 = 2s\tau^2 - \tau^4.]$$

Exercise 8.3.2. For the KE

$$\dot{x} = \cos\phi$$
$$\dot{y} = \sin\phi + 2\psi, \qquad -1 \leqslant \psi \leqslant 1$$

1. show analytically that the semipermeable surfaces are lines making a 30° angle with the vertical and find $\bar{\phi}$ and $\bar{\psi}$.
2. by drawing vectograms, demonstrate this result geometrically.

8.4. TERMINATION OF BARRIERS

It is quite possible for a semipermeable surface to come to an abrupt end, none of the paths being continuable beyond a certain curve in the surface so that the latter remains semipermeable.

That this phenomenon can occur is shown by three examples:

Example 8.4.1. Let the KE be

$$\dot{x} = \phi(1 - y) - (1 - y)$$
$$\dot{y} = -c\phi + c - 1, \qquad (c = \text{a constant};\ -1 \leqslant \phi \leqslant 1)$$

and we shall pass on semipermeable surface through (0, 0).

The ME is

$$\min\{\phi[\nu_1(1 - y) - \nu_2 c] - \nu_1(1 - y) + \nu_2(c - 1)\} = 0$$

so that

$$\bar{\phi} = -\sigma = -\text{sgn}[\quad] = -\text{sgn}\,A.$$

The RPE are

$$\mathring{x} = (\sigma + 1)(1 - y), \qquad \mathring{\nu}_1 = 0$$
$$\mathring{y} = -(\sigma + 1)c + 1, \qquad \mathring{\nu}_2 = (\sigma + 1)\nu_1.$$

As initial conditions we shall take $\nu_1 = 1$, $\nu_2 = 0$, as well as $x = 0$, $y = 0$. Such is consistent, for here

$$A = 1(1 - 0) - 0c = 1 > 0 \quad \text{and so } \sigma = -1, \bar{\phi} = 1$$

and the ME is satisfied.

As the RPE here are

$$\mathring{x} = 0, \qquad \mathring{\nu}_1 = 0$$
$$\mathring{y} = 1, \qquad \mathring{\nu}_2 = 0$$

they have the integrals

$$x = 0, \qquad \nu_1 = 1$$
$$y = \tau, \qquad \nu_2 = 0$$

and $A = 1 - \tau$.

Thus $A > 0$ until $\tau = 1$ and thus far the path is the vertical segment extending from (0, 0) to (0, 1). Here A changes sign and a continuation would have to have $\sigma = +1$, rendering the KE

$$\mathring{x} = 2(1 - y), \qquad \mathring{\nu}_1 = 0$$
$$\mathring{y} = 1 - 2c, \qquad \mathring{\nu}_2 = 2\nu_1 \tag{8.4.1}$$

Now if $c > \frac{1}{2}$ then $\mathring{y} = 1 - 2c < 0$ and the new path, if there is such, starts out downward, doubling back on the old. As the normal ν does not change at the junction, while the direction of traverse does, it follows that the orientation changes, and there can be no semipermeable continuation.

On the other hand, if $c < \frac{1}{2}$, the integrals of (8.4.1) present a perfectly valid continuation of our vertical segment. It is, as the reader may easily verify,

$$x = (1 - 2c)\tau^2$$
$$y = 1 + (1 - 2c)\tau \quad [6]$$

an arc of a parabola joining smoothly to the top of the vertical segment and forming with it a valid semipermeable surface.

It is instructive to work

Exercise 8.4.1. By drawing vectograms etc. interpret the above example geometrically.

A still simpler possibility is demonstrated by

Example 8.4.2. The KE being

$$\dot{x} = \sin\phi$$
$$\dot{y} = \cos\phi - q(x, y)$$

the vectograms appear as in Figure 8.4.1. If $q > 1$, either of the over-scored arrows (which one is a matter of orientation) are tangent to a semipermeable surface, as all the other arrows lie on the same side of these. On the other hand, if $q < 1$, clearly none can exist locally.

Let q, a function of x, y, be >1 in part of the plane and <1 in a second part. In the first part we can draw semipermeable surfaces. Should one of these tend to enter the second part, it must immediately terminate.

[6] Not the old τ; this one starts anew at the extension.

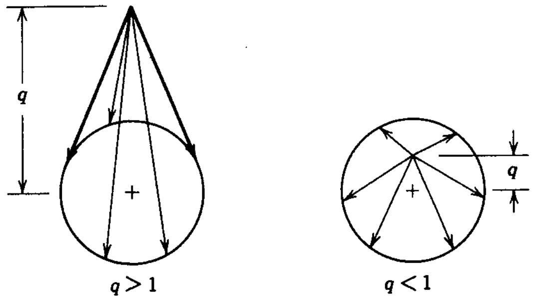


Figure 8.4.1

Exercise 8.4.2. By setting up the ME, RPE, etc., verify the preceding conclusions analytically. Construct and solve a specific case.

Finally, we display a typical instance of a proper two-person game. It will reappear later as a constituent of a broader matter in Chapter 10.

Example 8.4.3. Take $\mathscr{E}$ as the upper half-plane ($y \geqslant 0$) and let P have the vectogram XA_1A_2 shown at (a) of Figure 8.4.2. The headline A_1A_2 is vertical and its half height A_1A_3 or A_3A_2 is the constant b while the horizontal component XA_3 is $u(y)$. The circular vectogram of constant radius w belongs to E. The two velocities are additive so that the KE are

$$\dot{x} = u(y) + w \sin \psi$$
$$\dot{y} = -b\phi + w \cos \psi, \qquad -1 \leqslant \phi \leqslant 1.$$

Here $u(y)$ is positive, smooth, and increasing for $y \geqslant 0$. For some y_0, $w = u(y_0)$ so that $w \geqslant u$ in and only in the strip $0 \leqslant y < y_0$. Also $b > w$.

We will pass a semipermeable surface through the origin with the escape zone lying to the left. We shall soon see that it appears as at (b) of the figure, extending from 0 to a point B where $x\,(= x_B)$ is positive and $y\,(= y_B) = y_0$.

When $y < y_0$, draw the circle of center A_2 and radius w as shown at (c). We will demonstrate that the tangent XT_2 drawn as shown to this circle is the needed semipermeable direction at X. First let P play XA_2 ($\phi = 1$). Then the resultant velocities at E's disposal all begin at X and terminate on the circle; clearly none penetrates XT_2 in the direction of the arrow. On the other hand, if E plays A_2T_2, P can but choose velocity vectors extending from X to T_1T_2 (here A_1T_1 is a translate of A_2T_2) and none penetrates in the reverse direction. Hence the semipermeability.

Now suppose $y > y_0$ so that $u > w$. Then (see (d)) the line T_1T_2 lies on the same side of XT_2 as the circle and the semipermeability fails. In fact, no such direction exists with the proper orientation.

In the strip $0 \leqslant y < y_0$ we have an XT_2 through each point X and so a direction field. The classical theory of differential equations permits us to draw a curve through 0 having the local XT_2 as tangent at each of its points; its form is as at (b). Note that at B, the tangent is vertical, an obvious conclusion from the vector diagram (c).

Problem 8.4.1. Treat this example analytically and corroborate the foregoing results by the methods of Section 8.3.

Exercise 8.4.3. Show that the differential equation satisfied by the barrier is for $0 \leqslant y \leqslant y_0$

$$(b^2 - w^2)\frac{dx}{dy} = w\sqrt{u^2(y) + b^2 - w^2} - bu(y) \qquad (8.4.2)$$

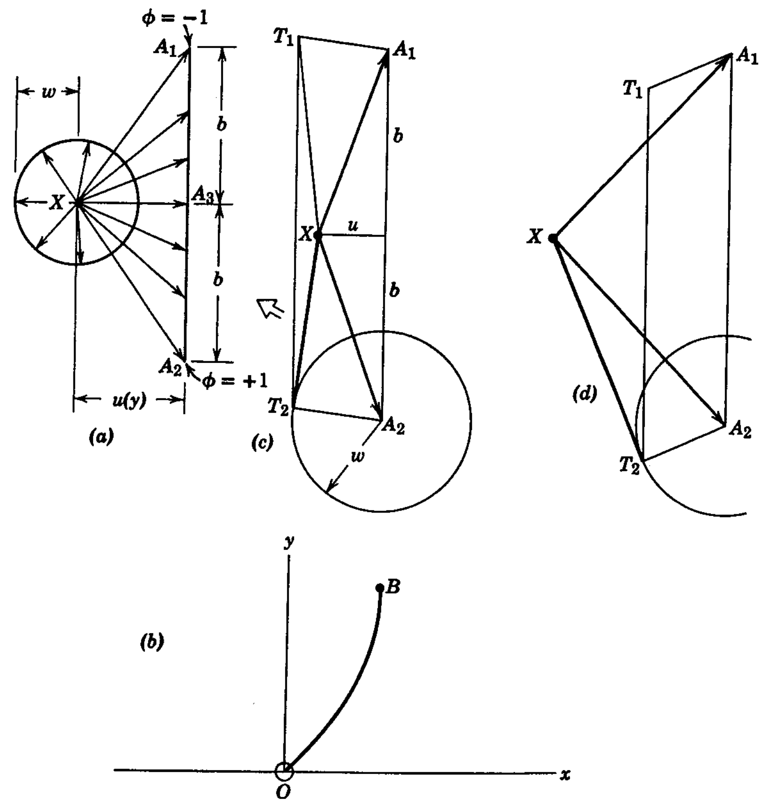


Figure 8.4.2

and obtain the barrier for the case[7] where

$$b = 3, \quad w = 2, \quad u = y + 1.$$

so that $y_0 = 1$.

[(8.4.2) can be derived easily by observing that normality requires

$$\frac{dx}{dy} = -\frac{\nu_2}{\nu_1}$$

and the ratio on the right can be found directly from the ME_2.]

8.5. CONSTRUCTION OF THE BARRIER

Our approach will be toward the delineation of the escape and capture zones, when both exist, by investigating the surface—the barrier—that separates them. We know it must be semipermeable. In the last section we saw how to construct such through a given curve. We now face the problem of ascertaining which curve.

From (8.3.1) we see that the normal vector ν to the barrier should extend *into* the escape zone. We will always adopt this orientation convention.

In many important cases the barrier will meet $\mathscr{C}$ and the initial curve can be taken as the intersection. This idea of taking $\mathscr{D}$ on $\mathscr{C}$ is in accord with our general scheme of beginning on $\mathscr{C}$ and working retrogressively into $\mathscr{E}$. That matters need not always be so is shown by

Example 8.5.1. We take as KE

$$\dot{x} = \cos\phi + u\cos\psi$$
$$\dot{y} = \sin\phi + u\sin\psi$$

where $u = u(x, y)$ is a continuous function enjoying the properties: For $y \geqslant 0$, $u > 0$. Letting $\mathscr{S}$ be a smooth curve in the upper half-plane ($y > 0$) which meets each vertical line exactly once. Above $\mathscr{S}$, $u > 1$; on $\mathscr{S}$, $u = 1$; below $\mathscr{S}$, $u < 1$.

We take $\mathscr{E}$ to be the upper half-plane (where $y \geqslant 0$) and $\mathscr{C}$ to be the x-axis.

It is clear that the velocity of **x** is vectorially the sum of two velocities, of magnitudes u and 1, with their respective directions under control of the players. Clearly, also, above $\mathscr{S}$, where $u > 1$, E controls the direction of motion and below $\mathscr{S}$, P does. Then $\mathscr{S}$ is the barrier. On it the two velocities each have magnitude 1; the players pull against one another, oppositely on the normal to $\mathscr{S}$. Consequently, $\mathscr{S}$ is static. Clearly the capture zone is the subset of below $\mathscr{S}$ and the escape is similarly above.

[7] We will utilize this case later.

Research Problem 8.5.1. The same as the last example except that a constant C is added to the expression for $\dot{x}$. If C is sufficiently small the barrier can be expected to be near $\mathscr{S}$. Is it still static?

From now on we assume $\mathscr{D}$ to lie on $\mathscr{C}$ and note three distinct possibilities.

I. *Natural Barriers.* Here $\mathscr{D}$ will be the boundary of the usable part. In the light of the present subject it is instructive to review the latter concept.

Let **x** be a point of $\mathscr{C}$ and let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be a nonzero vector at **x**, normal to $\mathscr{C}$ and extending into $\mathscr{E}$. The condition

$$\min_\phi \max_\psi \textstyle\sum_i \gamma_i f_i(\mathbf{x}, \phi, \psi) < 0, \qquad \mathbf{x} \in \mathscr{C} \tag{8.5.1}$$

expresses the fact that P can force **x** to penetrate $\mathscr{C}$ despite all opposition. The subset of $\mathscr{C}$ for which (8.5.1) is true is the useable part (of $\mathscr{C}$). If we replace $<$ by $>$, then E can frustrate penetration and such points constitutes the nonuseable part. The delineation—the BUP—is characterized by

$$\min_\phi \max_\psi \textstyle\sum_i \gamma_i f_i(\mathbf{x}, \phi, \psi) = 0, \qquad \mathbf{x} \in \mathscr{C}. \tag{8.5.2}$$

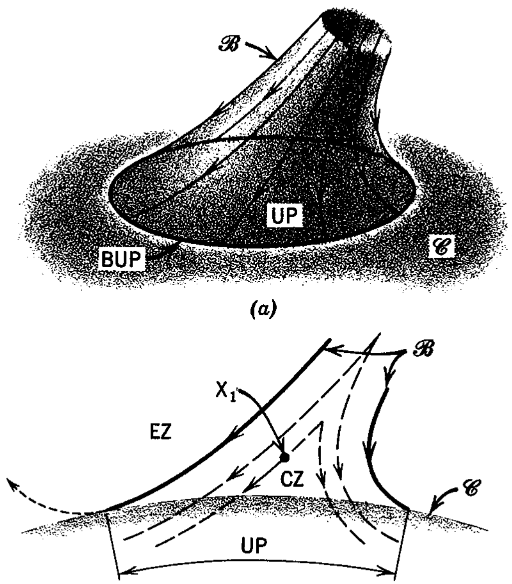
For such points, when each player exerts his optimal endeavor **x** moves (if at all) tangentially to $\mathscr{C}$.

As the BUP separates the points on $\mathscr{C}$ (or rather a little away from $\mathscr{C}$) where immediate capture will ensue from those with immediate escape, it seems logical to use the BUP as the initial curve $\mathscr{D}$ for the barrier $\mathscr{B}$.

To construct $\mathscr{B}$ we use for initial conditions: for **x**, the **x** of the BUP and for ν, we use γ. As γ is normal to $\mathscr{C}$, and coincides with ν where $\mathscr{B}$ meets $\mathscr{C}$, the two surfaces meet tangentially. An ideal prototype is sketched in Figure 8.5.1a. The BUP here is depicted as a closed curve on $\mathscr{C}$, the useable part being its interior. The barrier meets it tangentially to $\mathscr{C}$ and appears here as a trumpet shaped surface. It is the union of paths, whose directions are shown by arrows which meet $\mathscr{C}$ tangentially at the BUP and, as must be true generally, from the useable side.

Suppose now that $\mathscr{B}$ actually separates $\mathscr{E}$ into two parts. If **x** is in the outer side—the one not contiguous to the useable part—then P cannot compel capture. For he can force **x** neither through the semipermeable $\mathscr{B}$ nor through $\mathscr{C}$ as only the nonuseable part is accessible.

To show that the inner side of $\mathscr{B}$ is the capture zone is harder to prove. At (b) is shown a partial cross section of (a) of the figure, cutting the latter at a path. Let $\mathscr{B}$ be imbedded in a one-sided family of semipermeable surfaces as indicated by the dashed curves. These are assumed to bound ever smaller subsets of $\mathscr{E}$ over the useable part. Now let **x** be on the inner



(a)

(b)

Figure 8.5.1

side of $\mathscr{B}$, as X_1 is in the figure; X_1 lies on a surface of the family. If E plays a $\bar{\psi}$ germane to it, P will play $\bar{\phi}$; **x** stays on the path which leads him to and across $\mathscr{C}$. If E plays otherwise, then P can force him to penetrate through the family. This penetration is irreversible and—certainly in many particular cases at least—**x** is drawn irrevocably to $\mathscr{E}$.

Finally, let **x** be on $\mathscr{B}$. If $\bar{\phi}$ and $\bar{\psi}$ are played, **x** traverses a path of $\mathscr{B}$, meets $\mathscr{C}$ tangentially, and leaves it again (the dotted curve of (b)). Thus the outcome is neutral.

A defection by either player will lead to a worsened payoff, that is, either definite termination or escape. Thus here the optimal strategies of both players are really optimal in the customary sense of this term. Such is true nowhere else. For example, when outside of $\mathscr{B}$, E need adhere to no particular ψ. He can fix on some semipermeable surface neighboring $\mathscr{B}$ on the outside and arbitrarily close to it and not act decisively unless **x** reaches this surface.

We construct $\mathscr{B}$ by starting at the BUP and integrating the RPE therefrom. The resulting surface may or may not divide $\mathscr{E}$ into two parts. In the former case, these will be the sought escape and capture zones as just shown.

If $\mathscr{B}$ fails to subdivide $\mathscr{E}$, then capture can always be attained by P. But from starting points on the (in a local sense) two sides of $\mathscr{B}$ he must adopt different tactics. The typical idea is sketched in Figure 8.5.2. From X_2 we can expect rather a direct path to $\mathscr{C}$. But from X_1, P must force (assuming perhaps some reasonable resistance from E) **x** to follow the indirect path which skirts $\mathscr{B}$ to reach $\mathscr{C}$ at a point of the useable part. An example is the swerve maneuver in the homicidal chauffeur game, and there is a pristine one in the isotropic rocket game (see Figures 5.5.3, 5.5.4, and 5.5.5).

All portions of $\mathscr{B}$ need not be relevant to the problem at hand. Should $\mathscr{B}$ intersect itself, or consist of several parts which do, the portions beyond the intersection are to be discarded. Thus in Figure 8.5.3a, we reject the dashed parts of $\mathscr{B}$; the capture zone is the shaded curvilinear triangle. A beautiful instance of this, open to the most natural of interpretations, appears in the homicidal chauffeur game, to be discussed in Chapter 10.

Figure 8.5.3b depicts a discouraging possibility. Construction of the retrograde paths from some points of the BUP such as P_1, may be quite satisfactory. But from others the paths may dip below $\mathscr{C}$ when near the BUP and then rise above it, (P_2) piercing the useable part at a point A. Clearly such paths cannot fulfill the role we wish of them and must be discarded. An instance in the isotropic rocket game was for long the most frustrating problem encountered in the theory.

II. *Artificial Barriers*. Suppose we have a first game for which $\mathscr{C}$ has a useable part (which might be the whole of $\mathscr{C}$). The ultimate game is modified by adding the rule that termination must occur by **x** reaching $\mathscr{C}$ at a point within some definite subset of this part; this subset being bounded by a curve $\mathscr{D}$. To construct the barrier of the latter game, we

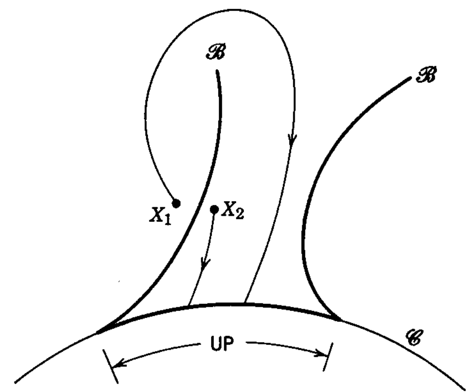
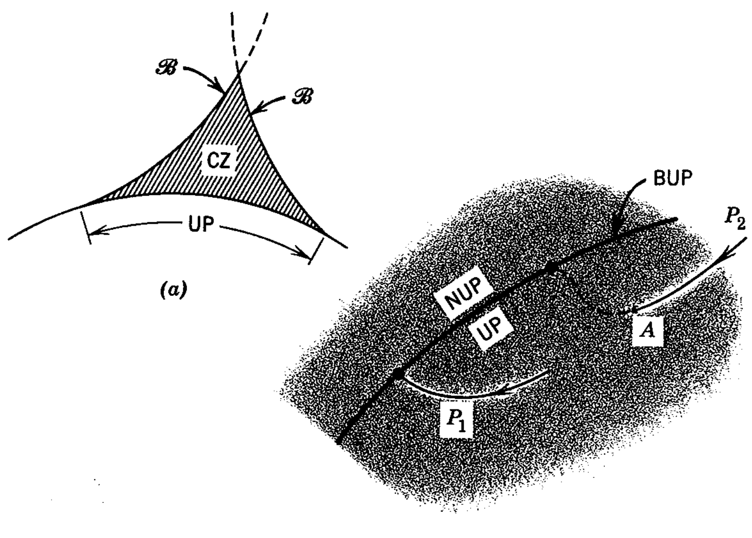


Figure 8.5.2



(a)

(b)

Figure 8.5.3

would begin by attempting to pass a semipermeable surface through $\mathscr{D}$. Such is one type of what we mean by an artificial $\mathscr{B}$.

Of course, the modification may not be naively stated as a curtailment of $\mathscr{C}$ but may be a logical equivalent. Thus if $\mathscr{C}$ were comprised of several analytic surfaces, $\mathscr{D}$ might lie at their intersections. A polygonal $\mathscr{C}$ is an example, only one face of which turns out to be the useable part.

A pursuit game with several pursuers hunting a single evader furnishes another instance. Another class restores the single pursuer but has the evader constrained by certain boundaries or obstacles. We can immediately consider the surfaces delineating these as additional surfaces of termination, regarding a transgression by $\mathbf{x}$ as tantamount to capture.

In all of these cases we begin by passing a semipermeable surface through $\mathscr{D}$ according to the methods of Section 8.3. Most of the preceding ideas of this section hold with evident modifications.

III. *Envelope Barriers.* Let G be a differential game with $n \geqslant 3$ and with a nonempty, nonuseable part of $\mathscr{C}$. The initial curve ($(n-2)$ manifold) $\mathscr{D}$ for this type of barrier lies in this part. The optimal paths emanating from $\mathscr{D}$ meet $\mathscr{D}$ tangentially (hence the term *envelope barrier*), and the optimal strategies on the barrier are extendable continuously onto $\mathscr{D}$. Thus when $\mathbf{x}$ follows an optimal path to $\mathscr{C}$, it can continue its motion, with both strategies continuous, and travel on $\mathscr{D}$.

Naturally only special $\mathscr{D}$ are eligible for the role and they do not exist in every game. Let $\mathbf{x}$ be in the nonuseable part. For any ϕ, then, E can find a ψ causing penetration *into* $\mathscr{E}$; let us suppose he can also find a $\psi = \breve{\psi} = \breve{\psi}(\mathbf{x}, \phi)$ which keeps $\mathbf{x}$ on $\mathscr{C}$, that is, its velocity is tangent to $\mathscr{C}$. Then when E plays $\breve{\psi}$, we have a one-player subgame G_1 whose $\mathscr{E}$ is (part of) the nonuseable part of G. The essential requirement on $\mathscr{D}$ is that it be a semipermeable surface of G_1.

We shall elucidate this in a theorem and then explain the role such surfaces play as barriers.

THEOREM 8.5.1. Let G be differential game with $n \geqslant 3$, for which the following is true:

1. There is a region $\mathscr{R}$ of the closed nonuseable part such that for any ϕ[8] and $\mathbf{x} \in \mathscr{R}$, there is a value $\breve{\psi}(\phi)$ ($= \breve{\psi}(\mathbf{x}, \phi)$) of ψ, which is continuous in $(\mathbf{x}, \phi)$, such that when ϕ and $\breve{\psi}(\phi)$ are played the resulting velocity vector does not penetrate $\mathscr{C}$. If more than one $\breve{\psi}$ is possible, we select some definite one.

2. In the one-player game, G_1 which has the KE

$$\dot{\mathbf{x}} = f(\mathbf{x}, \phi, \breve{\psi}(\phi))$$

and playing space $\mathscr{R}$, there is a semipermeable surface which we denote by $\mathscr{D}$.

3. In a neighborhood of $\mathscr{D}$, the optimal control variables are locally continuous with ψ locally unique. Explicitly these properties mean: For any fixed pair $\mathbf{x}$ of $\mathscr{D}$ and ν a normal to $\mathscr{D}$ there, we can select the minimizing ϕ of

$$Q(\phi, \psi) = \sum_i \nu_i f_i(\mathbf{x}, \phi, \psi)$$

so as to be continuous in $\mathbf{x}$ and ν in some neighborhood of the above fixed pair. For a definite ϕ, the corresponding maximizing ψ is a uniquely determined function of these variables and similarly continuous.

4. A semipermeable surface, $\mathscr{S}$, can be constructed in the standard way (Section 8.3) with $\mathscr{D}$ as initial curve.

Then $\mathscr{S}$ can be constructed so that its constituent optimal paths meet $\mathscr{D}$ tangentially and the values of the optimal ϕ and ψ for $\mathscr{S}$ agree with those of $\mathscr{D}$ at $\mathscr{D} \cap \mathscr{S}$.

Proof. We choose coordinates so that $\mathscr{C}$ is in the plane: $x_1 = 0$ with $\mathscr{E}$ on the side with $x_1 > 0$. Then for $\mathbf{x} \in \mathscr{R}$ and all ϕ

$$f_1(\mathbf{x}, \phi, \breve{\psi}(\phi)) = 0. \tag{8.5.3}$$

Let $\breve{\nu} = (0, \breve{\nu}_2, \ldots, \breve{\nu}_n)$ be the normal vector in $\mathscr{R}$ to $\mathscr{D}$.

[8] It is always understood that we deal only with admissible ϕ and ψ, that is, all constraints specified in the rules are always assumed satisfied.

By putting

$$Q_1(\phi, \psi) = \sum_{j=2}^{n} \breve{\nu}_j f_j(\mathbf{x}, \phi, \psi)$$

the semipermeable condition applied to $\mathscr{D}$ is

$$\min_{\phi} Q_1(\phi, \breve{\psi}(\phi)) = 0 = Q_1(\bar{\bar{\phi}}, \breve{\psi}(\bar{\bar{\phi}})) \tag{8.5.4}$$

where $\bar{\bar{\phi}} = \bar{\bar{\phi}}(\mathbf{x})$ supplies the min and $\mathbf{x} \in \mathscr{D}$.

The normal vector to $\mathscr{D}$ in $\mathscr{E}$, to be used as initial value for $\mathscr{S}$, is $(\nu_1, \breve{\nu}_2, \ldots, \breve{\nu}_n)$ for some ν_1. By putting

$$Q(\phi, \psi) = \nu_1 f_1(\mathbf{x}, \phi, \psi) + Q_1(\phi, \psi)$$

the semipermeable condition for $\mathscr{S}$ at $\mathscr{D}$ is

$$\min_{\phi} \max_{\psi} Q(\phi, \psi) = 0 = Q(\bar{\phi}, \bar{\psi}). \tag{8.5.5}$$

From (8.5.5)

$$Q(\bar{\phi}, \breve{\psi}(\bar{\phi})) \leqslant Q(\bar{\phi}, \bar{\psi}) = 0$$

and from (8.5.4)

$$Q_1(\bar{\phi}, \breve{\psi}(\bar{\phi})) \geqslant Q_1(\bar{\bar{\phi}}, \breve{\psi}(\bar{\bar{\phi}})) = 0 \tag{8.5.6}$$

If at least one of these inequalities were strict, we would have

$$\nu_1 f_1(\mathbf{x}, \bar{\phi}, \breve{\psi}(\bar{\phi})) = Q(\bar{\phi}, \breve{\psi}(\bar{\phi})) - Q_1(\bar{\phi}, \breve{\psi}(\bar{\phi})) < 0$$

contradicting (8.5.3).

Therefore, both of (8.5.6) are equalities. From the second, when we construct $\mathscr{D}$, we can use $\bar{\bar{\phi}}$ in place of $\bar{\phi}$, as hypothesis 3 justifies. From the first, and again from 3, the $\breve{\psi}(\bar{\phi})$ on $\mathscr{D}$ agrees with $\bar{\psi}$ on $\mathscr{S}$. The optimal strategies for $\mathscr{S}$ and $\mathscr{D}$ are equal at their juncture. Because this implies that the $\dot{x}_i$ are also equal there, we have the stipulated tangency.

Remarks. If more than one optimizing ϕ or ψ is possible at $\mathscr{D}$, we may be able to construct more than one semipermeable surface through $\mathscr{D}$. Only one, $\mathscr{S}$, of this necessarily discrete set can have the continuous junction just exhibited.

If more than one $\breve{\psi}$ is possible there will be an $\mathscr{S}$ for each. The proper choice of $\breve{\psi}$ is dictated by the exigencies of the particular problem as will be seen in the examples of Chapter 9.

The hypothesis 3 is necessary. We can utilize Exercise 8.3.1 as a counterexample (we adjoin -2 and 2 as bounds for ψ). It is clear that we may take $\breve{\psi}$ as $-\sqrt{x_2}$. Calculation, with $\nu_3 = 1$, yields for $\mathscr{D}$: $x_1 = 0$, $0 < x_2 \leqslant 1$, $x_3 = 0$. But the reader who has solved the exercise will know that on $\mathscr{S}$, $\bar{\psi} = \nu_2/\nu_1$ and that this ratio approaches 0 as we near $\mathscr{D}$. This discontinuity in the optimal ψ precludes tangency of the paths to $\mathscr{D}$ and the conclusion of the theorem is untrue.

Observe that $\mathscr{S}$ with its boundary $\mathscr{D}$ together constitute a semipermeable surface.

COROLLARY 8.5.1. If $\mathscr{D}$ meets the BUP, the optimal paths of $\mathscr{D}$ meet it tangentially.

Proof. In the coordinates of the last proof, if $\mathbf{x} \in$ BUP then

$$\min_{\phi} \max_{\psi} f_1(\mathbf{x}, \phi, \psi) = 0$$

and let $\bar{\phi}$ and $\bar{\psi}$ now supply the minimax here. By 3. we may suppose (after choosing a "branch," if necessary) them unique near a juncture of $\mathscr{D}$ and the BUP. At a point K of this juncture $\bar{\phi}$ and $\bar{\psi}$ must thus be $\bar{\bar{\phi}}$ and $\breve{\psi}(\bar{\bar{\phi}})$.

Near K on the BUP we then have definite $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$. If they are used as control variables, then $\mathbf{x}$ must traverse a path in the BUP. The velocity vector $(0, \dot{x}_2, \ldots, \dot{x}_n)$ of this path at K is the same as that of $\mathscr{D}$ there, as the control variables agree. Hence tangency.

How are such $\mathscr{S}$ to serve as the barrier of G? First $\mathscr{D}$ must meet the BUP. Then if $\mathbf{x}$ starts from $\mathscr{S}$, under optimal play it first traverses its path in $\mathscr{S}$, then one in $\mathscr{D}$ until it reaches the BUP. Here we may expect that $\mathbf{x}$ will escape into $\mathscr{E}$ in the manner (the dotted curve) of Figure 8.5.1b. Thus until escape $\mathbf{x}$ has remained on a semipermeable surface and the outcome is neutral.

By virtue of Corollary 8.5.1, $\mathscr{D}$ and the BUP have a common normal $\boldsymbol{\nu}$ when they meet. Thus the paths constituting $\mathscr{S}$ must blend smoothly into those of the natural barrier. In such a case, then, the natural and envelope barriers together constitute one composite barrier.

Observe that at a point of the juncture, the mutual optimal path, as one of $\mathscr{S}$, is tangent to $\mathscr{D}$ and so by the corollary tangent to the BUP. Thus only through points of a natural barrier at which the barrier paths are tangent to the BUP is it possible to construct a $\mathscr{D}$.

Due to this latter tangency it may well be that envelope barriers are the remedy to the disconcerting phenomenon depicted in Figure 8.5.3b. We know this is so in one case.[9]

Research Problem 8.5.2. In Figure 8.5.3b, let K be the point on the BUP which delineates the paths that dip under from those that do not. Is it generally possible to pass a $\mathscr{D}$ through K and construct an envelope barrier which takes over from the valid paths of the natural barrier?

There is one last but important difficulty. When $\mathbf{x}$ is on $\mathscr{D}$ (which $\in \mathscr{C}$)

[9] The isotropic rocket (Example 9.3).

it may be quite possible for P to cause penetration of $\mathscr{C}$. In the language of the theorem's proof, P has a determined optimal $\bar{\bar{\phi}}$[10] when $\mathbf{x}$ is traversing $\mathscr{D}$. But this $\bar{\bar{\phi}}$ may not be the minimizing one for $f_1(\mathbf{x}, \phi, \psi)$. Then penetration certainly can be forced.

Further insight requires

COROLLARY 8.5.2. Suppose $\mathscr{S}$ actually separates the escape and capture zones of G. If there is a value ϕ_1[11], of ϕ such that, for $\mathbf{x} \in \mathscr{D}$

$$f_1(\mathbf{x}, \phi_1, \breve{\psi}(\bar{\bar{\phi}})) < 0$$

then there is a value ψ_1 of ψ such that, when ϕ_1, ψ_1 are played, $\mathbf{x}$ will penetrate both $\mathscr{C}$ and $\mathscr{S}$ into the escape zone.

Proof. As ϕ_1 is not optimal for the semipermeable surface, the use of $\breve{\psi}(\phi_1)$ by E will cause $\mathbf{x}$ to penetrate $\mathscr{D}$ and such must be toward the escape side. The velocity $\dot{\mathbf{x}}$ arising, being of G_1, is tangent to $\mathscr{C}$. As $\mathscr{D}$ is in the nonuseable part, E will certainly have a ψ_2 which leads to penetration of $\mathscr{C}$ into $\mathscr{E}$ when opposed by ϕ_1. By the Convexity Assumption (Section 2.7) any convex linear combination of $\breve{\psi}(\phi_1)$ and ψ_2 will be in E's vectogram. If he chooses one with the lion's share of the former and sufficiently small coefficient for the latter, the resultant ψ_1 will have the properties stated.

Thus, should P attempt a capture with ϕ_1 while $\mathbf{x}$ is on $\mathscr{D}$, E can retaliate by escaping with ψ_1. But this seems to require decisions based on the opponent's control variable, in violation of the definition of a strategy. Technically we can reply that definite $\bar{\phi}$ and $\bar{\psi}$ exist at all points of the barrier; if the players navigate with them there is no hitch.

Put practically, E must have some foil for the above nonoptimal capture threat. Let him play in accordance a $\mathscr{C}_\varepsilon$ replacing $\mathscr{C}$ as described in the "assuagement" near the end of Section 8.2. He responds, should P pull $\mathbf{x}$ beneath $\mathscr{C}_\varepsilon$, but before, say, it reaches $\mathscr{C}_{\varepsilon/2}$. The simplest way seems to permit the above violation, our excuse being the short duration of the lapse. He meets ϕ_1 with ψ_1 and, with a small enough ε, gets into the escape zone in a very short time.

Very likely this defect can be ameliorated, but the small interim of the unorthodoxy seems to occasion little practical difficulty In some cases at least, E could act as if ϕ_1 were extreme (minimizes f_1) and define a responsive strategy by defining $\bar{\psi}$ as the ψ_1, countering this extreme ϕ for $\mathbf{x}$ in a lamina below $\mathscr{C}_\varepsilon$.

[10] Possibly a vacuous statement, for f_1 may be independent of ϕ. Then, of course, there is no possible penetration and no difficulty.

[11] Subscripts, in this proof, do not mean components.

8.6. SOME BRIEF EXAMPLES

Example 8.6.1. Interception of a straight flying evader. Here P moves in the plane with simple motion and unit speed, while E is bound to a line (say, the x-axis), moves with speed w, and merely can select for his strategy one of the two possible directions of travel. Capture occurs when $|PE| \leqslant l$.

In view of P's unquestioned ability to capture when $w < 1$, this case is trivial. Our interest is in the conditions which make possible the success of a slower pursuer ($w \geqslant 1$).

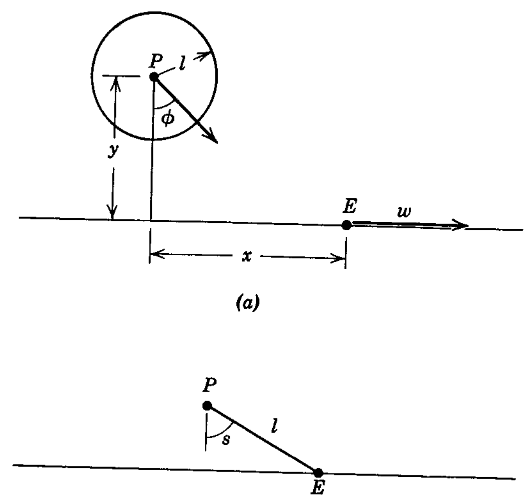
The ideas here, although possibly not this very simple exemplification, pertain to a practically important question: When can an interceptor be successful against a faster attacking craft which travels a fixed straight course? The latter might be a ballistic missile with no provision for evasive maneuver or an attacking aircraft whose speed or plan permits little or no such.

As we shall see shortly, this problem is very simple. We give the full formal solution to illustrate our context.

Let coordinates be as in Figure 8.6.1a so that the KE are

$$\dot{x} = w\psi - \sin\phi$$
$$\dot{y} = -\cos\phi$$

where $-1 \leqslant \psi \leqslant 1$ (preferable to $\psi = \pm 1$).



(a)

(b)

Figure 8.6.1

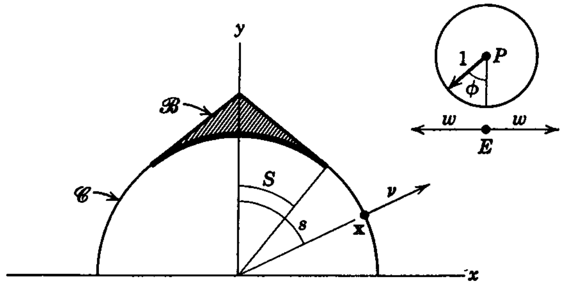


Figure 8.6.2

The ME_1 is

$$\min_{\phi}\max_{\psi}\,[-(\nu_1 \sin\phi + \nu_2 \cos\phi) + w\nu_1\psi] = 0$$

and so $\sin\bar{\phi} = \nu_1/\rho$, $\cos\bar{\phi} = \nu_2/\rho$ with $\rho = \sqrt{\nu_1^2 + \nu_2^2}$ and

$$\bar{\psi} = \sigma = \operatorname{sgn}\nu_1.$$

Thus the ME_2 is

$$-\rho + w\nu_1\sigma = 0$$

and the RPE

$$\mathring{x} = -w\sigma + \frac{\nu_1}{\rho}, \qquad \mathring{\nu}_1 = \mathring{\nu}_2 = 0$$

$$\mathring{y} = \frac{\nu_2}{\rho}.$$

Capture occurs ((b) of the figure) when

$$x = l \sin s$$
$$y = l \cos s.$$

In the reduced space we have the semicircle of Figure 8.6.2 as $\mathscr{C}$. We prefer to consider only $y \geqslant 0$. The useable part is defined by (note the vectograms sketched and that ν, the normal, $= (\sin s, \cos s)$)

$$\min_{\phi}\max_{\psi}\,[-(\sin\phi \sin s + \cos\phi \cos s) + w\psi \sin s] \leqslant 0$$

or

$$-1 + w\,|\sin s| \leqslant 0$$

or

$$|\sin s| \leqslant \frac{1}{w}.$$

That is, if S is defined by $\sin S = 1/w$, $0 \leqslant S \leqslant \pi/2$, then the useable part of $\mathscr{C}$ is

$$-S \leqslant s \leqslant S \quad \text{(the heavy arc in the figure)}$$

and so the BUP is given by

$$s = \sigma S. \quad (\text{recall } \sigma = \operatorname{sgn}\nu_1).$$

Thus the initial conditions are

$$x = \sigma l \sin S, \qquad \nu_1 = \sigma \sin S$$
$$y = l \cos S \qquad \nu_2 = \cos S.$$

The path integrals (of the RPE) are quickly obtained:

$$x = \sigma[(l + \tau)\sin S - w\tau]$$
$$y = (l + \tau)\cos S.$$

The slope here is

$$\sigma \frac{\cos S}{\sin S - w}$$

which, because $\sin S = 1/w$, is equal to

$$-\sigma \frac{\sin S}{\cos S}$$

and hence the barriers are tangent to $\mathscr{C}$ at the points where $s = \sigma S$. We curtail them at their intersection (if $w > 1$). Thus we reach the result that the capture zone is the shaded region of Figure 8.6.2.

The formal work could have been shortened. By observing that x and y do not appear on the right in the KE, we infer that $\mathring{\nu}_i = 0$ and then that the barriers must be straight. Hence, as soon as we know the useable part, we can draw the properly directed tangents from its endpoints at once.

Note that if $w = 1$, the barriers do not meet. In this case, the capture zone is the strip

$$|x| < l$$

The barriers are the lines $x = \pm l$ and are static.

Problem 8.6.1. Confirm the above results geometrically when $w > 1$.
Problem 8.6.2. Discuss the "realistic" significance of the result when $w = 1$.

To treat the interception problem mentioned earlier, we merely deprive E of his ability to change his direction of motion. Only one barrier now appears and the situation is shown in Figure 8.6.3.

Example 8.6.2. Dresher's one chance pursuit game. We reconsider Example 6.6.1, but we will generalize slightly by permitting motion with arbitrary speeds. Letting the vectograms be as Figure 8.6.4a, and x and y

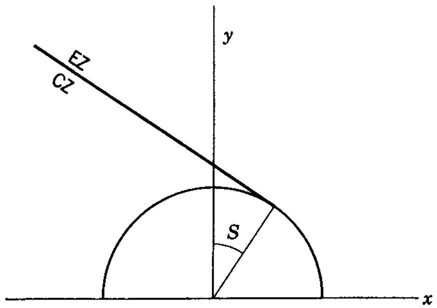


Figure 8.6.3

being the components of the vector PE, the KE are

$$\dot{x} = w\psi + \phi$$
$$\dot{y} = -(w + 1), \qquad w \gtreqless 1.$$

Capture as usual means $x^2 + y^2 < l^2$. In the reduced space $\mathscr{C}$ is parameterized by s as in Figure 8.6.2 except now we admit the full circle instead of the upper half. The useable part is identified by

$$\min_{\phi} \max_{\psi} [(w\psi + \phi) \sin s - (w + 1) \cos s]$$
$$= \sigma(w - 1) \sin s - (w + 1) \cos s < 0$$

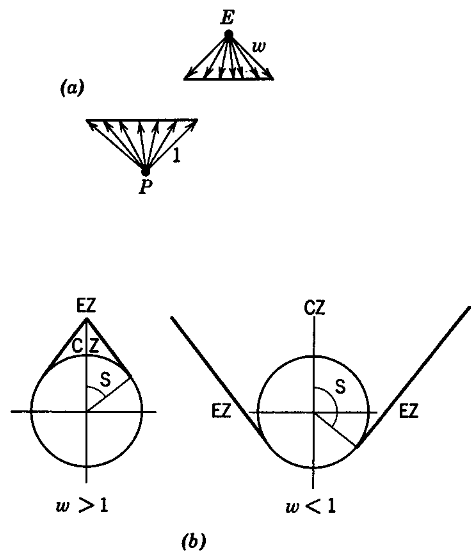


Figure 8.6.4

where $\sigma = \operatorname{sgn} \sin s$. Thus the critical angle S, connoting the BUP satisfies

$$\tan S = \frac{w + 1}{w - 1}.$$

We can employ the short-cut of the previous example: the right sides of the KE, being free of x and y, lead to straight barriers. They appear as in (b) of Figure 8.6.4.

In the case treated in Example 6.6.1, $w = 1$ and the barriers then appear vertical and parallel.

Example 8.6.3. A synthetic problem. Using the KE of Exercise 8.3.1, take $\mathscr{E}$ as $x_3 \geqslant 0$ and $\mathscr{C}$ as $x_3 = 0$. The useable part is decided by

$$\min_{\phi} \dot{x}_3 = \min_{\phi} (-x_1 + 1 - \phi) = -x_1 < 0$$

or $x_1 > 0$. The BUP is thus the $\mathscr{D}$, the x_2-axis of Exercise 8.3.1, and we refer to the solution obtained there:

$$x_1 = 4s\tau - 4\tau^3$$
$$x_2 = s - 2\tau^2 \qquad (8.6.1)$$
$$x_3 = 2s\tau^2 - \tau^4.$$

For $s < 0$, observe that $x_3 < 0$ for small positive τ and thus these paths cannot form part of the barrier. We thus suppose that $s \geqslant 0$.

When $\tau = \sqrt{2s}$, the path pertaining to s returns to $x_3 = 0$ and it meets this plane on the curve

$$x_1 = -4\sqrt{2}\, s^{\frac{3}{2}}$$
$$x_2 = -3s, \qquad 0 \leqslant s < \infty.$$

Given any $x_3 > 0$ and x_2, we shall show that there exists uniquely s, τ with $s > 0$, $0 < \tau < \sqrt{2s}$ such that the last two of (8.6.1) are satisfied. Thus the barrier meets every parallel in $\mathscr{E}$ to the x_1-axis exactly once and so divides $\mathscr{E}$ into two parts.

The proof involves merely some routine elementary algebra. Solving the last two of (8.6.1) for s and τ yields for the latter

$$\tau^2 = \frac{-x_2 \pm \sqrt{x_2^2 + 3x_3}}{3}.$$

By choosing $\pm = +$ we get just one possible positive τ. Then

$$s = x_2 + 2\tau^2 > 0.$$

From

$$x_3 = (2s - \tau^2)\tau^2$$

follows that

$$\tau < \sqrt{2s}.$$

Example 8.6.4. The dolichobrachistochrone. We have seen in Example 5.2 that for starting points low in the plane—when $y < w^2$—E can prevent termination despite all efforts of P. Thus the line $y = w^2$ appears to be a barrier. Like a natural barrier it is a semipermeable surface through the BUP, but it is *not* tangent to $\mathscr{C}$. This exceptional behavior is caused by the barrier's being static.

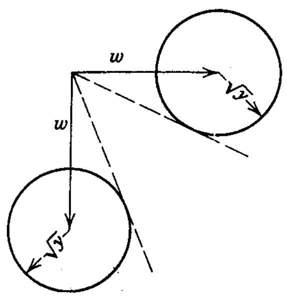


Figure 8.6.5

Exercise 8.6.1. Prove that line $y = w^2$ is actually semipermeable, static, and properly oriented.

Problem 8.6.3. Show that above this line ($y > w^2$) no semipermeable surface exists. Show that below it two pass through each point. This can be easily done geometrically as suggested by Figure 8.6.5. Finally, explain why the juncture of the transition surface and BUP confounds the tangency of $\mathscr{C}$ and $\mathscr{B}$.

8.7. POSSIBLE OTHER SPECIES OF BARRIERS

Our classification of barriers arose from an exploration of methodology and does not attempt exhaustion. That still further varieties may arise is indicated by

Example 8.7.1. A further type of barrier. In this planar one-player game the KE are

$$\dot{x} = w \cos \psi$$

$$\dot{y} = w \sin \psi - u(x)$$

where $u(x)$ is a smooth function enjoying the properties:

$$u(x) > 0 \text{ and } u'(x) < 0 \text{ for all } x.$$

$$u(0) = w.$$

It follows $u(x) > w$ when $x < 0$ and $u(x) < w$ when $x > 0$.
For $\mathscr{C}$ we take the line through the origin

$$x = s, \qquad y = \alpha s$$

where α is a fixed number of either sign; $\mathscr{E}$ lies above $\mathscr{C}$.

We are going to state some results, leaving the proofs to the reader.
Let A be the point of $\mathscr{C}$ (see Figure 8.7.1a) such that

$$\sqrt{u^2(A) - w^2}/w = |\alpha|$$

so that $s < 0$. The useable part of $\mathscr{C}$ lies to the left of A regardless of the sign of α.

When $x < 0$ the family $\mathscr{F}$ of curves which are the integrals of

$$\frac{dy}{dx} = \sqrt{u(x)^2 - w^2}/w$$

are semipermeable. (They are sketched at (a) of the figure.)

When $\alpha < 0$, one member $\mathscr{B}$ of $\mathscr{F}$ will be tangent to $\mathscr{C}$ at A. It constitutes a perfectly normal natural barrier (see (b) of the figure).

But when $\alpha > 0$, the barrier $\mathscr{B}$ is that member of $\mathscr{F}$ which passes through the origin O (see (c)). It is not tangent to $\mathscr{C}$ and meets it at the nonuseable part. For starting points just above AO, such as the X shown, capture does not occur immediately, but X follows a route as indicated to the useable part.

Exercise 8.7.1. Attain these conclusions analytically.

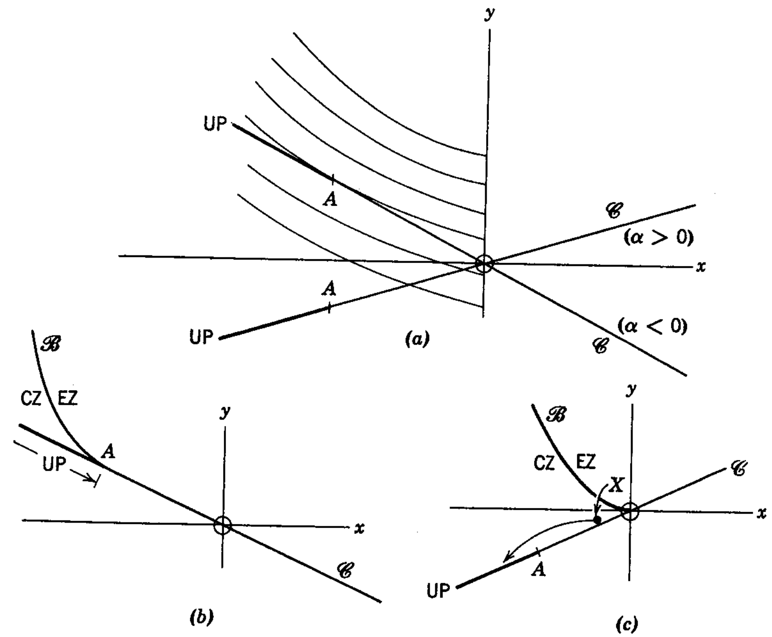


Figure 8.7.1

8.8. FUSION OF GAMES OF KIND AND DEGREE

We have already discussed one aspect of this question in an early chapter. In a pursuit game, when the starting position is in the capture zone, P might adopt the attitude: "As I can capture, I shall do so as efficiently as possible and (say) minimize the time it takes." If E correspondingly seeks to maximize, we have a composite game of kind and degree.

But there is a second possibility of fusing the two kinds of games. For $\mathbf{x}$ in the escape zone, P might say, "I cannot attain capture, but I will come as close to it as I can; I shall (say) minimize my closest distance[12] to E."[13]

In both aspects there arises the question of the similarity between the optimal strategies (for both players) of degree and kind. The latter will be defined on the barrier; the former in a half-neighborhood of it at least. The two $\bar{\phi}(\mathbf{x})$ $[\bar{\psi}(\mathbf{x})]$ thus defined can be regarded as one function on the closure of the half-neighborhood. If this function is continuous, the two types of strategies can be said to fuse continuously with one another.

Whether or not there occurs this fusion is closely related to another matter, also of interest in its own right:

THE ENVELOPE PRINCIPLE. If a pursuit game is solved with time of capture as the payoff, the envelope of the surfaces of constant V is the barrier.

A beautiful instance of the principle's holding is depicted in Figure 5.5.4, where the envelope of the circular arcs of constant V is strikingly visible and does constitute a cross section of the barrier. In the homicidal chauffeur game, the principle holds for small V, but fails for larger, as we shall learn in Chapter 10.

Lack of space necessitates postponement of publication of our researches into these questions.

[12] The mean distance is another possibility.

[13] He may be mentally adding, "Who knows? That fool of an E may not play optimally and I'll still get him."

CHAPTER 9

Examples of Games of Kind

The ideas of the preceding chapter are applied to a number of examples. The homicidal chauffeur game and its smoother counterpart, the isotropic rocket, are both solved in full. The game of two cars (Section 9.2) illustrates the complexities that may be submerged in apparently innocent problems. Its ideas here prove to be of practical importance in their similarity to those of collision avoidance.

What we have called the lifeline and deadline games are a pair with the same kinematic background and apparently analogous content. But their solutions exhibit a remarkable distinction.

The former game is a skeletal prototype of an evading craft trying to reach the (straight) boundary of large target area prior to capture by a faster intercepter.

A naval interceptor (torpedo, ship, etc.) trying to nail a faster evader with a shoreline at its back is a broad instance of the deadline game.

Variations of this game (Section 9.6) are rich in basic applications which, although simple in principle, are deceptively complex of solution. In the one-sided version the evader strives to pass between shoreline and pursuer. This leads to the cornered rat and corridor games. In the former, say, the evader is trapped in a "bay" and endeavors to slip past the pursuer hovering near its mouth. Still retaining naval terms, the latter game has the evader trying to slip past a pursuer in a channel or river (or it might be a football player passing a single tackler). We obtain the important critical channel width.

This quantity also applies to the spacing of a patrol line of craft endeavoring to frustrate passage of a faster evader. A similar idea governs a circular patrol line seeking to prevent the escape of a centrally located and faster evader.

In Section 9.7 we offer some suggestions as the applicability of our ideas to von Neumann's general bounded pursuit games.

We conclude by exhibiting but not exemplifying a method of solving dogfight (such as between aircraft with nonaimable weapons) games.

There is also a strong possibility of applying these ideas to the theory of stability, but details, we feel, lie outside our compass. Such are sketched in the final section.

9.1.[1] THE HOMICIDAL CHAUFFEUR GAME

We shall now investigate the conditions under which the more agile but slower E can avoid being run over by the faster but curvature-bound P.

The variables, already discussed, are indicated in Figure 9.1.1. Again we encounter the KE

$$\dot{x} = -\frac{w_1}{R}\, y\phi + w_2 \sin \psi$$

$$\dot{y} = \frac{w_1}{R}\, x\phi - w_1 + w_2 \cos \psi, \qquad -1 \leqslant \phi \leqslant 1$$

which lead to the ME_1:

$$\min_{\phi}\max_{\psi}\left\{-\frac{w_1}{R}[y\nu_1 - x\nu_2]\phi - w_1\nu_2 + w_2(\nu_1 \sin\psi + \nu_2 \cos\psi)\right\} = 0.$$

If we put

$$A = y\nu_1 - x\nu_2$$

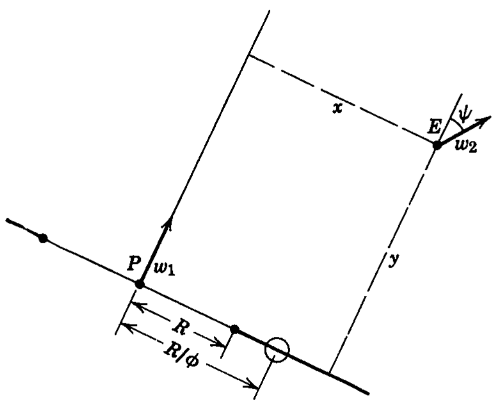


Figure 9.1.1

[1] In much of this chapter, sections and examples will coincide.

then
$$\bar{\phi} = \operatorname{sgn} A = \sigma$$
and if
$$\rho = \sqrt{\nu_1^2 + \nu_2^2}$$
then
$$\cos\bar{\psi} = \frac{\nu_2}{\rho}, \qquad \sin\bar{\psi} = \frac{\nu_1}{\rho}$$
so that the ME_2 is

$$-\sigma\frac{w_1}{R}A - w_1\nu_2 + w_2\rho = 0.$$

The RPE are found to be

$$\mathring{x} = cy - w_2\frac{\nu_1}{\rho}, \qquad \mathring{\nu}_1 = c\nu_2$$

$$\mathring{y} = -cx + w_1 - w_2\frac{\nu_2}{\rho}, \qquad \mathring{\nu}_2 = -c\nu_1$$

where
$$c = \frac{w_1}{R}\sigma.$$

Turning now to the initial conditions, $\mathscr{C}$ is defined by

$$x = l \sin s, \qquad y = l \cos s$$

and the outward normal γ by

$$\gamma_1 = \sin s, \qquad \gamma_2 = \cos s.$$

Thus the useable part is decided by

$$\begin{aligned}\min_{\phi}\max_{\psi}[\gamma_1\dot{x} + \gamma_2\dot{y}] &= \min\max\left[\sin s\left(-\frac{w_1}{R}(l\cos s)\phi + w_2\sin\psi\right)\right. \\ &\quad \left. + \cos s\left(\frac{w_1}{R}(l \sin s)\phi - w_1 + w_2\cos\psi\right)\right] \\ &= \max_{\psi}[-w_1\cos s + w_2\cos(\psi - s)] \\ &= w_2 - w_1 \cos s \leqslant 0.\end{aligned}$$

Defining S by

$$0 \leqslant S \leqslant \frac{\pi}{2}, \qquad \cos S = \frac{w_2}{w_1}$$

the useable part is thus specified by

$$|s| < S$$

and its boundary, the BUP, by $s = \pm S$.

We note that on $\mathscr{C}$

$$A = (l\cos s)\sin s - (l \sin s)\cos s = 0$$

so that we look to $\overset{\circ}{A}$, which is (in all of $\mathscr{E}$ as well)

$$\overset{\circ}{A} = y(c\nu_2) - x(-c\nu_1) + \nu_1\left(-cx + w_1 - w_2\frac{\nu_2}{\rho}\right) - \nu_2\left(cy - w_2\frac{\nu_1}{\rho}\right)$$
$$= w_1\nu_1. \tag{9.1.1}$$

Thus, on $\mathscr{C}$, $\sigma = \operatorname{sgn} \nu_1 = \operatorname{sgn} s$. We will work with the right barrier, taking

$$\sigma = 1.$$

The left-hand side is, of course, fully symmetric. The initial conditions used for the integration of the RPE are thus

$$x = l \sin S$$
$$y = l \cos S = l\frac{w_2}{w_1}$$
$$\nu_1 = \sin S$$
$$\nu_2 = \cos S.$$

Integrating the last two RPE gives

$$\nu_1 = \sin(S + c\tau)$$
$$\nu_2 = \cos(S + c\tau), \qquad c = \frac{w_1}{R} \tag{9.1.2}$$

so that the first two are

$$\overset{\circ}{x} = cy \qquad -w_2 \sin(S + c\tau)$$
$$\overset{\circ}{y} = -cx + w_1 - w_2 \cos(S + c\tau) \tag{9.1.3}$$

which have, as may be readily checked, the integrals

$$x = (l - w_2\tau)\sin(S + c\tau) + R(1 - \cos c\tau)$$
$$y = (l - w_2\tau)\cos(S + c\tau) + R \sin c\tau. \tag{9.1.4}$$

Let us define the circles $\mathscr{K}_+$ and $\mathscr{K}_-$ as being concentric with the minimal turning circles and having radii in the speed ratio to them. That is, $\mathscr{K}_\pm$ has center at $(\pm R, 0)$ and radius $(w_2/w_1)R$.

The barrier is the involute of $\mathscr{K}_+$ which is tangent to $\mathscr{C}$ as shown in Figure 9.1.2a for (9.1.4) are the equations of just this curve.[2]

We have, either directly or by integrating (9.1.1)

$$A = R[\cos S - \cos(S + c\tau)]$$

so that, on $\mathscr{B}$, A ceases to be positive when $S + c\tau = 2\pi - S$. Here, from (9.1.2), the normal to the curve is $(-\sin S, \cos S)$. It is not hard to see that this normal must lie along the line OB, which is the lower tangent

[2] In Chapter 10 we shall rediscover this involute by purely geometric methods.

from the origin to $\mathscr{K}_+$. If we attempted to prolong $\mathscr{B}$ further, we would have to change σ to -1. The extension would have to be an arc of an involute of $\mathscr{K}_-$; everything jibes here, for clearly the normal OB is tangent to both $\mathscr{K}_+$ and $\mathscr{K}_-$. But the new involute would have to unwind counterclockwise from $\mathscr{K}_-$ and there would result a doubling back of $\mathscr{B}$, an absurdity. Therefore $\mathscr{B}$ terminates, and the full (right hand) barrier is the arc of the involute extending from C to B, which is overscored in the figure.

There is, of course, a symmetric curve on the left. The two may or may not meet, depending on the parameters, as shown in (c) and (b) of Figure 9.1.2. In the latter case ((b)), $\mathscr{E}$ is not divided into two parts by $\mathscr{B}$, and

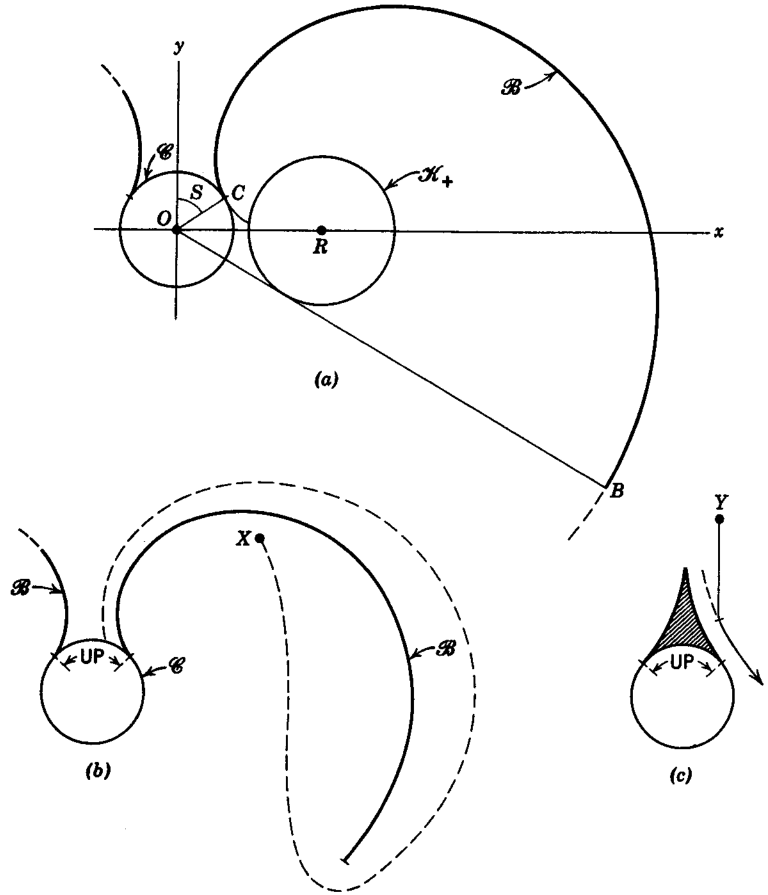


Figure 9.1.2

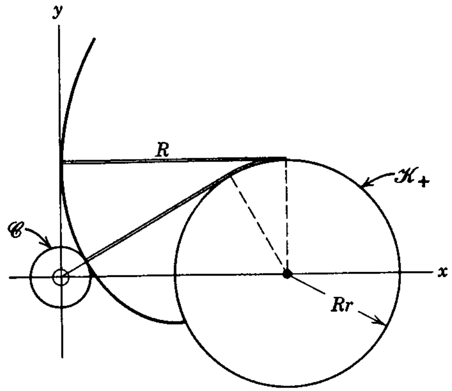


Figure 9.1.3

consequently all of $\mathscr{E}$ is the capture zone. That is, P can always achieve capture. From starting points such as X, E can force P into employing the devious route shown around $\mathscr{B}$; such is what happens under a swerve (Section 1.5). Thus $\mathscr{B}$, although it does not delineate capture and escape, does demarcate the starting points from which optimal play will entail a swerve.

When the involutes meet as at (c), we erase the parts of them beyond their intersection. The shaded curvilinear triangle is the capture zone; all the exterior region is the escape. We interpret the situation heuristically. Suppose the parameters are very much in E's favor, that is, P's margin of speed is not great, l is small, and the turning radius large. The most natural way for E to avoid being hit in such circumstances is simply to sidestep whenever capture seems imminent. From a starting position such as Y, it matters not what E does immediately. We can think of him as, say, remaining stationary. An attempted hit by P will bring $\mathbf{x}$ downward from Y; E is not obliged to act until $\mathbf{x}$ is near $\mathscr{B}$, when he sidesteps. This would, most naturally, mean that $\mathbf{x}$ traverses an involute near and external to $\mathscr{B}$ as shown at (c). He has sidestepped; the imminent danger is past until P aligns himself for a second pass at E, whence the same type of thing recurs.

The shaded capture zone consists of those positions where E is placed so closely in front of the advancing P that, despite the former's kinematic advantages, he cannot sidestep. As it is confined to a bounded set of starting points, while the escape zone is a set of infinite extent, we feel justified stating that whether or not the involutes meet is essentially the criterion for capture or escape.

With the aid of Figure 9.1.3 (equate the doubly drawn lengths), it is an elementary matter to write an analytic criterion. If $\gamma =$ the speed

ratio $= w_2/w_1 < 1$, then capture occurs if

$$\boxed{\frac{l}{R} > \sqrt{1-\gamma^2} + \sin^{-1}\gamma - 1} \qquad (9.1.4)$$

and escape if this inequality is reversed.

For capture regions of other, even asymmetric, shapes than circular, the same type of criterion applies: whether or not the contacting involutes of $\mathscr{K}_+$ and $\mathscr{K}_-$ meet.[3]

9.1A. DOGFIGHTING A HIGHLY MOBILE TARGET

For a craft, such as a single seater airplane with fixed machine guns, let the lethal area covered by the guns be of a form such as is sketched in Figure 9.1.4. Their fixed orientation relative to the craft, together with some stricture on motion such as bounded curvature; causes their effectiveness to be greatly dependent on maneuver.

We can take the aircraft as P in the foregoing game, with the lethal area as capture region, and E the target. We could learn, say, how fast the target must be in order to be able to remain out of gun range.

This problem will usually be of greatest interest when E has a speed greater than P and such necessitates extending the foregoing analysis.

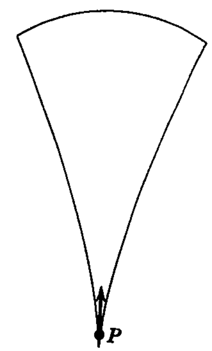


Figure 9.1.4

9.2. THE GAME OF TWO CARS

In this and the following section we take up two problems of the same type as, but more difficult than, the homicidal chauffeur game. The present one leads to an analysis so formidable that we must be content with a solution complete in principle, but not fully displayed. Of course, any particular case—an individual set of parameter values—could be computed to completion.

The two car game was the subject of Example 8.1.1. It is just like the homicidal chauffeur except the evader too suffers the constraint of bounded curvature. Let w_1 and w_2 be the speeds of P and E and R_1 and R_2 their respective minimal radii of curvature. The dimensionality of the reduced space is 3, and there are many ways of selecting coordinates. We have chosen x, y, θ as clarified by Figure 9.2.1. These appear to be practical from the standpoint of handling the somewhat oppressive differential

[3] For sufficiently large regions, of course, no barriers will exist. For example of the capture circle contains $\mathscr{K}_+$ and $\mathscr{K}_-$. Then all of $\mathscr{E}$ is the capture zone.

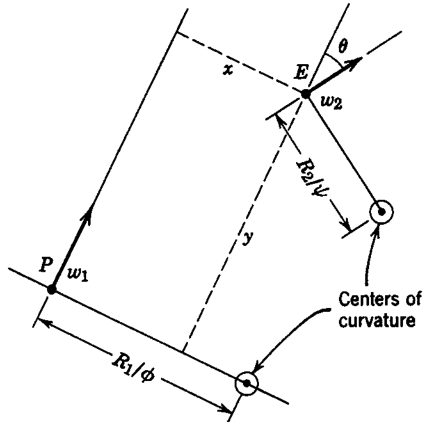


Figure 9.2.1

equations as well as being similar enough to the homicidal chauffeur game to employ generalizations. Figure 9.2.2 depicts $\mathscr{E}$, the reduced space; it is exterior to the cylinder of radius l. Of course, θ is toroidal; it extends from 0 to 2π, and the planes of these two values are to be thought of as coinciding.

It is not hard to write the KE:

$$\dot{x} = -\frac{w_1}{R_1} y\phi + w_2 \sin\theta$$

$$\dot{y} = \frac{w_1}{R_1} x\phi - w_1 + w_2 \cos\theta$$

$$\dot{\theta} = -\frac{w_1}{R_1}\phi + \frac{w_2}{R_2}\psi, \quad -1 < \phi, \psi < 1.$$

Let us put

$$A = \nu_1 y - \nu_2 x + \nu_3.$$

The ME_1 is then

$$\min_{\phi}\max_{\psi}\left[-\phi\frac{w_1}{R_1}A + w_2(\nu_1\sin\theta + \nu_2\cos\theta) + \frac{w_2}{R_2}\nu_3\psi - w_1\nu_2\right] = 0.$$

Thus

$$\bar{\phi} = \sigma_1 = \operatorname{sgn} A$$

$$\bar{\psi} = \sigma_2 = \operatorname{sgn} \nu_3.$$

The RPE are found by our usual methods. To abbreviate we put

$$c = \frac{w_1}{R_1}\sigma_1$$

the RPE being

$$\mathring{x} = cy - w_2\sin\theta, \qquad \mathring{\nu}_1 = c\nu_2$$

$$\mathring{y} = -cx + w_1 - w_2\cos\theta, \qquad \mathring{\nu}_2 = -c\nu_1$$

$$\mathring{\theta} = c - \frac{w_2}{R_2}\sigma_2, \qquad \mathring{\nu}_3 = w_2(\nu_1\cos\theta - \nu_2\sin\theta).$$

We can also calculate that

$$\mathring{A} = w_1\nu_1. \tag{9.2.1}$$

We now turn to the initial conditions. We parametrize $\mathscr{C}$ by

$$x = l\sin s_1$$

$$y = l\cos s_1$$

$$\theta = s_2$$

as shown in Figure 9.2.2. If $r^2 = x^2 + y^2$ we have from the KE

$$r\dot{r} = x\dot{x} + y\dot{y} = x(w_2\sin\theta) + y(-w_1 + w_2\cos\theta)$$

which on $\mathscr{C}$ becomes

$$l\dot{r} = l\sin s_1(w_2\sin s_2) - l\cos s_1(w_1 - w_2\cos s_2).$$

Putting

$$W = \sqrt{w_1^2 + w_2^2 - 2w_1w_2\cos s_2}$$

we see that the boundary of the useable part (where $\dot{r} = 0$) is given by

$$\sin s_1 = \pm\frac{w_1 - w_2\cos s_2}{W}, \qquad \cos s_1 = \pm\frac{w_2\sin s_2}{W}. \tag{9.2.2}$$

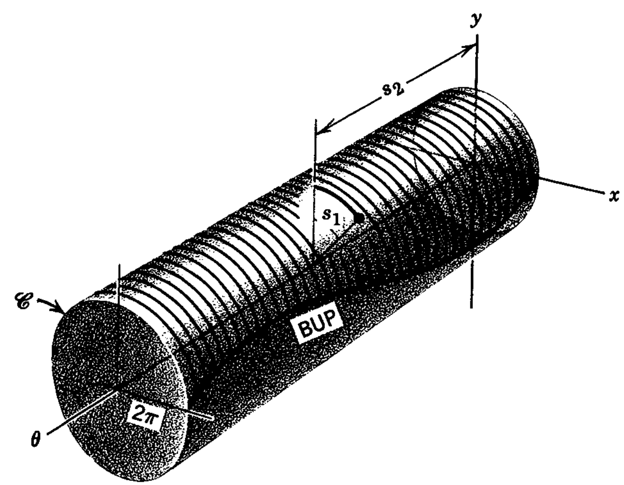


Figure 9.2.2

The $\pm$ means that, on each section of $\mathscr{C}$ where $\theta = s_2 =$ constant, the BUP consists of a pair of diametrically opposite points.

We now must make a distinction as to whether $w_1 > w_2$ or the reverse. In the former case, we see from (9.2.2) that $\sin s_1$ never changes sign. Thus the two BUP remain more or less on opposite sides of $\mathscr{C}$ and the useable part roughly spans the upper half of $\mathscr{C}$. Such is crudely shown in Figure 9.2.2. On the other hand, if $w_1 < w_2$, the sign change in s_1 (when $\cos s_2 = w_1/w_2$) means that the useable spirals around $\mathscr{C}$.

Exercise 9.2.1. By means of vector diagrams interpret these conclusions geometrically.

To save space we shall treat only the case where $w_1 > w_2$ in detail. To ascertain $\bar{\phi}$ and $\bar{\psi}$ at the outset, we must look to A and ν_3 on the BUP. As here

$$\nu_1 = \sin s_1 = \pm \frac{w_1 - w_2 \cos s_2}{W}$$

$$\nu_2 = \cos s_1 = \pm \frac{w_2 \sin s_2}{W}$$

$$\nu_3 = 0$$

we see readily that $A = \nu_3 = 0$ on the BUP. Thus the criteria are the signs of $\mathring{A}$ and $\mathring{\nu}_3$.

Hence from (9.2.1)

$$\sigma_1 = \operatorname{sgn} \mathring{A} = \operatorname{sgn} \nu_1 = \operatorname{sgn}\left(\pm \frac{w_1 - w_2 \cos s_2}{W}\right) \tag{9.2.3}$$

$$= (\text{assuming } w_1 > w_2) \pm 1.$$

Thus $\sigma_1 = +1$ throughout the right-hand BUP. By symmetry we may work with this alone and from now on take $\sigma_1 = 1$ and $\pm = +$.

From the RPE

$$\begin{aligned} \sigma_2 &= \operatorname{sgn} \mathring{\nu}_3 = \operatorname{sgn}(\nu_1 \cos \theta - \nu_2 \sin \theta) \\ &= \operatorname{sgn}[w_1 \cos s_2 - w_2]. \end{aligned} \tag{9.2.4}$$

This changes sign, again supposing $w_1 > w_2$. Let S be the angle in the first quadrant such that

$$\cos S = \frac{w_2}{w_1}.$$

Therefore, in the intervals of s_2 below, σ_2 has the values:

0 to S,	$\sigma_2 = 1$
S to $2\pi - S$,	$\sigma_2 = -1$
$2\pi - S$ to 2π,	$\sigma_2 = 1.$

Let us glance at the RPE. Clearly $\mathring{x}$ and $\mathring{y}$ do not depend on σ_2, but $\mathring{\theta}$ does and so changes abruptly when $s_2 = S$ and $2\pi - S$. As s_2 increases through S, $\mathring{\theta}$ suffers a sudden increase. This means that the paths from $s_2 = S-$ and $s_2 = S+$ diverge from one another. We have a void, and it must be filled by a universal curve and its tributary paths.

At $2\pi - S$, on the other hand, $\mathring{\theta}$ suddenly decreases; the surfaces formed by the paths from either side of $2\pi - S$ will intersect. They must be curtailed where they do so and here will be a dispersal curve. Note that as $\bar{\phi} = 1$ throughout this right-hand barrier, no instantaneous mixed strategy is necessary.

Let us turn our attention to the universal curve. As it is a ψ-UC, in the notation of Chapter 7, we write the α_i and β_i from the KE and compute the γ_i:

i	α_i	β_i	γ_i
1	0	$-cy + w_2 \sin\theta$	$(-w_2 \cos\theta)w_2/R_2$
2	0	$cx - w_1 + w_2 \cos\theta$	$(w_2 \sin\theta)w_2/R_2$
3	w_2/R_2	$-c$	0

Suppressing the factors w_2/R_2 and w_2^2/R_2, the determinant is

$$c(-y \sin\theta + x \cos\theta) - w_1 \cos\theta + w_2 = 0. \tag{9.2.5}$$

Which curve on this surface? Differentiate (9.2.5) with respect to τ, and use the RPE (with $\breve{\psi}$ replacing σ_2) to find the navigable paths which should comprise the surface.

$$\begin{aligned} 0 &= c[-\mathring{y} \sin\theta + \mathring{x} \cos\theta] + [c(-y\cos\theta - x\sin\theta) + w_1 \sin\theta]\mathring{\theta} \\ &= c[c(y\cos\theta + x\sin\theta) - w_1 \sin\theta] + [\text{as above}]\left(c - \frac{w_2}{R_2}\breve{\psi}\right) \\ &= [c(y\cos\theta + x\sin\theta) - w_1\sin\theta]\frac{w_2}{R_2}\breve{\psi}. \end{aligned}$$

Thus either $\breve{\psi} = 0$ or the [ ] = 0. The latter implies, together with (9.2.5),

$$cx - w_1 + w_2 \cos\theta = 0$$
$$cy - w_2 \sin\theta = 0.$$

These two equations belong to a curve in $\mathscr{E}$ on which, as we see from the RPE, $\mathring{x} = \mathring{y} = 0$. Such a static configuration hardly suits our end. Therefore we take $\breve{\psi} = 0$. Turning again to the RPE we have, after replacing σ_1 by 1 and σ_2 by 0,

$$\begin{aligned} \mathring{x} &= cy - w_2 \sin\theta \\ \mathring{y} &= -cx + w_1 - w_2\cos\theta \\ \mathring{\theta} &= c. \end{aligned}$$

From the third equation $\theta = S + c\tau$. Making this replacement in the first two, we see that they agree exactly with (9.1.3), the barrier equations of the homicidal chauffeur problem. Thus the projection of our universal curve on the x, y-plane is the involute barrier of that problem.

This conclusion is quite harmonious with our general conception of a universal surface. It means that E, when playing the optimal neutral strategy, executes a sharp left or right turn until his orientation is suitable for him to play just as in the homicidal chauffeur game.

We have not yet integrated the RPE. Doing so, with the initial conditions for x_i and ν_i already found, we obtain the result below, which can be readily checked. We retain σ_1 and σ_2 so that these paths fit all aspects of our problem.

$$x = l \sin (s_1 + c\tau) + R_1\sigma_1(1 - \cos c\tau) + R_2\sigma_2(\cos (s_2 + c\tau) - \cos \theta)$$

$$y = l \cos (s_1 + c\tau) + R_1\sigma_1 \sin c\tau - R_2\sigma_2(\sin (s_2 + c\tau) - \sin \theta) \tag{9.2.6}$$

$$\theta = s_2 + \left(c - \frac{w_2}{R_2}\sigma_2\right)\tau$$

$$\nu_1 = \sin (s_1 + c\tau)$$

$$\nu_2 = \cos (s_1 + c\tau)$$

$$\nu_3 = R_2\sigma_2 \left[\cos (s_1 - s_2) - \cos \left(s_1 - s_2 + \frac{w_2}{R_2}\sigma_2\tau\right)\right].$$

In the above it is understood that s_1 is to be replaced by its value (9.2.2). We adjoin to the above

$$A = R_1\sigma_1(\cos s_1 - \cos (s_1 + c\tau)).$$

We are interested in the smallest positive τ, called τ_A and τ_3, which annul A and ν_3. Without bothering with details we presume that the smaller of such τ marks the termination of the barrier. We have

$$\tau_A = \frac{R_1}{w_1} 2(\pi - s_1) \qquad \text{if} \quad \sigma_1 = +1$$

$$= \frac{R_1}{w_1} 2s_1 \qquad \text{if} \quad \sigma_1 = -1$$

$$\tau_3 = \frac{R_2}{w_2} 2(\pi - (s_1 - s_2)) \qquad \text{if} \quad \sigma_2 = +1$$

$$= \frac{R_2}{w_2} 2(s_1 - s_2) \qquad \text{if} \quad \sigma_2 = -1.$$

Figure 9.2.3 is a very crude attempt to indicate in part the appearance of the right barrier. The left one will be similar; note that on it, the dispersal curve (DC) emerges from $s_2 = 2\pi - S$ and the universal (UC) from $s_2 = S$, and that the former curve moves toward rather than away from the plane of $\theta = 0$.

The escape criterion is the meeting of these two surfaces and their bounding, together with the useable part of $\mathscr{C}$, a portion of $\mathscr{E}$. But how can we tell if such happens? It seems a formidable computation indeed.

Even the calculation of the dispersal curve is frustrating. The principle nevertheless is clear. In the three equations (9.2.6) we put $\sigma_1 = 1$ (if we are working with the right barrier) and replace s_1 by its value from (9.2.2) with $\pm = +$. We write the equations twice, with $\sigma_2 = +$ and -1 and consider τ and s_2 distinct in each of the two sets. Then we equate the two x, y, and θ. There are then three equations in the four unknowns consisting of the two τ and the two s_2. They should have a one-parameter family of solutions which define the dispersal curve we seek. Of course, the τ obtained must be between 0 and min $\{\tau_A, \tau_3\}$ and the two s_2 must be larger and smaller than $2\pi - S$.

For any particular set of w_1, w_2, R_1, R_2, l[4] we could arrive at the empirical answer. One way would be to plot cross sections of constant θ of the two barriers and see whether or not they all meet. The necessary equations are obtained easily from a minor manipulation of (9.2.6).

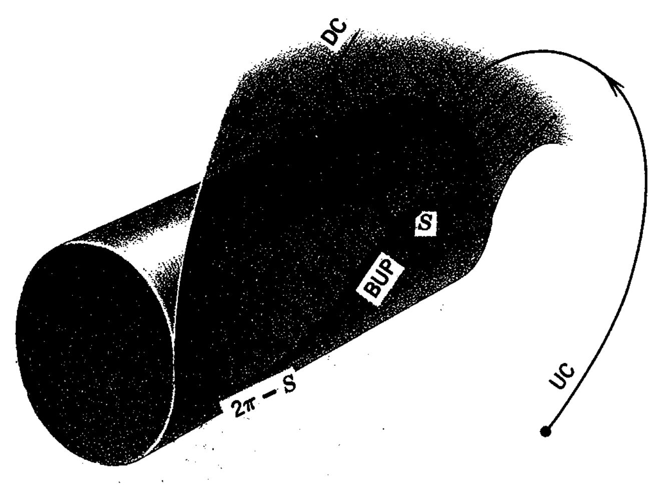


Figure 9.2.3

[4] As ratios only of speeds and distances matter, the effective number of parameters is at most three (for example, w_1/w_2, R_1/l, R_2/l).

What changes if we treat the case of $w_1 < w_2$? From (9.2.2) s_1 and hence σ_1 changes sign when $\cos s_2 = w_1/w_2$, but from (9.2.4) σ_2 now remains constant on each barrier. It seems that the barriers will be broken by roughly the same kind of dispersal and universal curves as before, but now the latter will be a ϕ-UC instead of a ψ-UC. We might conclude that it is always the player with the slower speed whose strategy is beset with discontinuities of singular surfaces, *regardless of the curvature radii.*

Research Problem 9.2.1. Investigate the case of $w_1 < w_2$ more fully. In particular, what is the UC? Is it the same as the barrier in a pursuit game in which P has simple motion and E bounded curvature?

9.3. THE ISOTROPIC ROCKET

We return to this problem, solved with time of capture as payoff in Section 5.5, and construct the barrier. This time a precise escape criterion can be found, although sharp proofs for a few details appear to be recalcitrant. We adopt the reduced coordinates x, y, v as before (see Figure 9.3.1a), although we replace the X and Y by small letters. We also drop the friction force. Even though unbounded speeds for P are thus admitted, the problem gains in simplicity of formal mathematics without any sacrifice of principle. We will indicate later what changes result from a restoration of friction.

The KE, readily derived, are

$$\dot{x} = -F\frac{y}{v}\sin\phi + w\sin\psi$$

$$\dot{y} = F\frac{x}{v}\sin\phi + w\cos\psi - v$$

$$\dot{v} = F\cos\phi.$$

The reduced space (x, y, v with $x^2 + y^2 \geqslant l^2, v \geqslant 0$) $\mathscr{E}$ is shown in (b) of the figure. As usual, l is the radius of the capture circle. We parametrize $\mathscr{C}$ by $x = l\sin s_1$, $y = l\cos s_1$, $v = s_2$.

Putting $r = \sqrt{x^2 + y^2}$ we compute the useable part by

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= x\left(-F\frac{y}{v}\sin\phi + w\sin\psi\right) + y\left(F\frac{x}{v}\sin\phi + w\cos\psi - v\right) \\ &= w(x\sin\psi - y\cos\psi) - vy. \end{aligned}$$

Thus, on $\mathscr{C}$, for the boundary of the useable part

$$\max_{\psi} l\dot{r} = l(w - s_2\cos s_1) = 0. \tag{9.3.1}$$

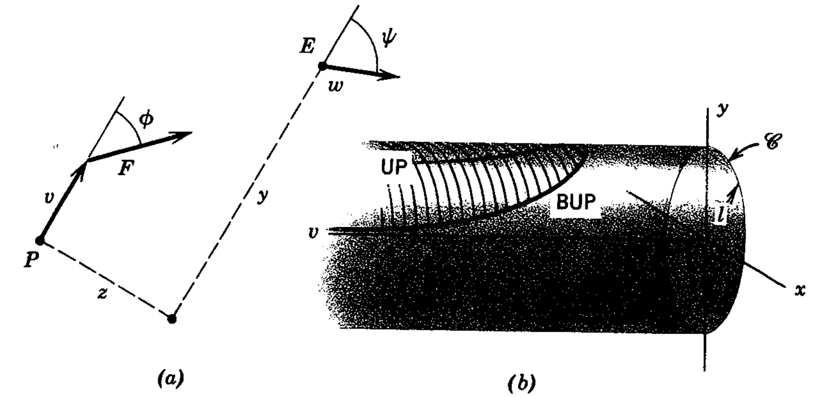


Figure 9.3.1

Looking along the x-axis we will thus see the BUP as the cylinder cut by the hyperbola

$$vy = wl.$$

(See (b) of the figure.)

Using $s_2 = s$ as parameter we can write for the BUP, using (9.3.1),

$$\begin{aligned} x &= \pm l\sqrt{1 - (w/s)^2} \\ y &= l\frac{w}{s} \\ v &= s, \qquad s \geqslant w. \end{aligned} \tag{9.3.2}$$

The rest of the initial conditions are, as is easily inferred from ν's being normal to (9.3.2) and satisfying the ME_2 below:

$$\nu_1 = \pm\sqrt{1 - (w/s)^2}$$

$$\nu_2 = \frac{w}{s}$$

$$\nu_3 = 0.$$

The ME_1 is

$$\max_{\psi}\min_{\phi}\left[F\left(\frac{-U}{v}\sin\phi + \nu_3\cos\phi\right) + w(\nu_1\sin\psi + \nu_2\cos\psi) - v\nu_2\right] = 0$$

where $U = \nu_1 y - \nu_2 x$.

If we also put

$$\rho_1 = \sqrt{U^2/v^2 + \nu_3^2}, \qquad \rho_2 = \sqrt{\nu_1^2 + \nu_2^2}$$

we have

$$\sin \bar{\phi} = \frac{U}{v\rho_1}, \qquad \cos \bar{\phi} = -\frac{\nu_3}{\rho_1}$$

$$\sin \bar{\psi} = \frac{\nu_1}{\rho_2}, \qquad \cos \psi = \frac{\nu_2}{\rho_2}$$

and the ME_2 is

$$-F\rho_1 + w\rho_2 - v\nu_2 = 0.$$

We are led to the RPE

$$\mathring{x} = F\frac{yU}{v^2\rho_1} - w\frac{\nu_1}{\rho_2}, \qquad \mathring{\nu}_1 = F\frac{\nu_2 U}{v^2\rho_1}$$

$$\mathring{y} = -F\frac{xU}{v^2\rho_1} - w\frac{\nu_2}{\rho_2} + v, \qquad \mathring{\nu}_2 = -F\frac{\nu_1 U}{v^2\rho_1}$$

$$\mathring{v} = F\frac{\nu_3}{\rho_1}, \qquad \mathring{\nu}_3 = F\frac{U^2}{v^3\rho_1} - \nu_2.$$

The closed integration of this system, with the foregoing initial conditions, presents its elementary difficulties, but the result is (we use as a convenient abbreviation $W = F\tau - w$):

$$x = \frac{\pm\sqrt{s^2 - w^2}}{v}\left[l - w\tau + \frac{1}{2}F\tau^2\right], \qquad \nu_1 = \frac{\pm\sqrt{s^2 - w^2}}{v}$$

$$y = \frac{1}{v}\left[\left(\frac{1}{2}F\tau^2 - l\right)W + (s^2 - w^2)\tau\right], \qquad \nu_2 = -\frac{W}{v} \tag{9.3.3}$$

$$v = \sqrt{s^2 - w^2 + W^2}, \qquad \nu_3 = \frac{W\tau}{v}.$$

The $\pm$ distinguishes the right ($x > 0$) and left sides of $\mathscr{E}$. By symmetry, we may restrict ourselves to $\pm = +$, that is, the right barrier.

We also compute from the above

$$r^2 = x^2 + y^2 = l^2 + (s^2 - w^2 - lF)\tau^2 + \frac{1}{4}F^2\tau^4. \tag{9.3.4}$$

As we should expect $r(0) = l$, $\mathring{r}(0) = 0$. But we also observe that when

$$s^2 - w^2 - Fl < 0$$

or when

$$s < S = \sqrt{w^2 + Fl} \tag{9.3.5}$$

then $\mathring{\mathring{r}}(0) < 0$.
Thus when (9.3.5) obtains, our paths enter $\mathscr{C}$ for small τ and so are useless for constituents of a barrier.[5] Let us call the points of the BUP which

[5] An instance of the phenomenon mentioned in Section 8.5 (see Figure 8.5.3).

mark the beginning of this phenomenon B^+ and B^-. They have the coordinates

$$B^{\pm}: \; x = \pm l\frac{\sqrt{S^2 - w^2}}{S} = \pm l\frac{\sqrt{Fl}}{S}$$

$$y = l\frac{w}{S}$$

$$v = S.$$

Our next result is: at $B^{\pm}$, the path is tangent to the BUP.

For the tangent direction to the BUP is obtained by differentiating (9.3.2) and putting $s = S$.

$$x_s = l\frac{w^2}{S^2\sqrt{Fl}}, \quad y_s = -l\frac{w}{S^2}, \quad v_s = 1.$$

Also at $B^{\pm}$

$$\mathring{x} = -\sqrt{Fl}\frac{w^3}{S^3}, \quad \mathring{y} = \frac{Flw^2}{S^3}, \quad \mathring{v} = -F\frac{w}{S}$$

and these two triples are one a multiple of the other.

The escape condition is that the two parts of the barrier meet or that there exist $\tau_0 = \tau_0(s) > 0$ such that $x(\tau_0) = 0$. From (9.3.3) this means that

$$F\tau^2 - 2w\tau + 2l$$

have a positive root. Thus the discriminant $w^2 - 2Fl$ must be positive. This condition is sufficient, for we can take for τ_0 the lower root:

$$\tau_0 = \frac{w - \sqrt{w^2 - 2Fl}}{F} \tag{9.3.6}$$

which is plainly positive.

The above suggests that

$$\boxed{\text{the escape condition is } w^2 \geqslant 2Fl} \tag{9.3.7}$$

but substantiation is needed.

When (9.3.7) holds, let us call the curve where the barriers meet the *crest*. Its form is interesting. If we substitute the value (9.3.6) of τ_0 for τ into the y and v of (9.3.3) we find, after a little manipulation that

$$y = v\tau_0. \tag{9.3.8}$$

Thus the crest is a straight line through the origin, in the ($x = 0$)-plane, with a slope (y/v) given by (9.3.6).

When (9.3.7) holds, our work thus far leads to a barrier having the appearance of Figure 9.3.2.

The barrier, as thus far constructed, has the form of a semi-infinite tapering tent. But the right (on the figure) end is open, for beyond B_+ and B_- our barrier construction fails. What sort of semipermeable surface will seal it? This question proved the most difficult one thus far met in the subject. The answer will be given shortly.

The remarks about sidestepping, in the discussion of the homicidal chauffeur game, apply here. There is no need to reiterate the argument that a meeting of the barriers implies possible escape for E.

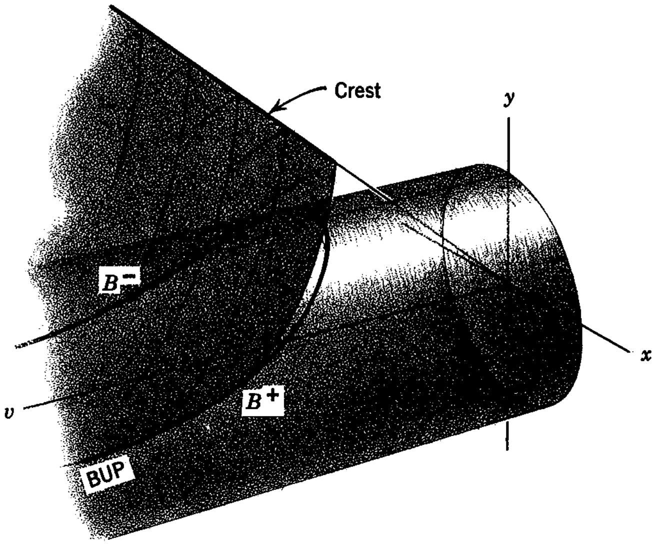


Figure 9.3.2

If friction were restored, we recall that our function $Q(\tau)$ (see Section 5.5) is the radius of the cylinder of which the surface of constant τ is part. The envelope principle (Section 8.8) holds; the barriers are the envelopes of these cylinders. Thus the meeting of the barriers is equivalent to the radius somewhere being 0. Therefore we can generalize the condition (9.3.7) by replacing it by

$$\min_{\tau} Q(\tau) = 0.$$

Exercise 9.3.1. Show that, with friction, the critical condition which demarcates capture and escape is

$$\left(\frac{F}{wk}\right)\left[1 - \exp\left(-\frac{k(w - lk)}{F - wk}\right)\right] = 1. \qquad (9.3.9)$$

Replacing = by < signifies capture.

Show also that as $k \to 0$, this condition reduces to the present case:

$$w^2 = 2Fl.$$

Research Problem 9.3.1. We have said nothing about the termination of the barriers. When the escape condition (9.3.7) does not obtain, we should expect the barriers to demarcate the necessity of something analogous to the swerve maneuver; indeed Figure 5.5.5 bears this out. Thus it is reasonable to expect that the barriers do come to an end.

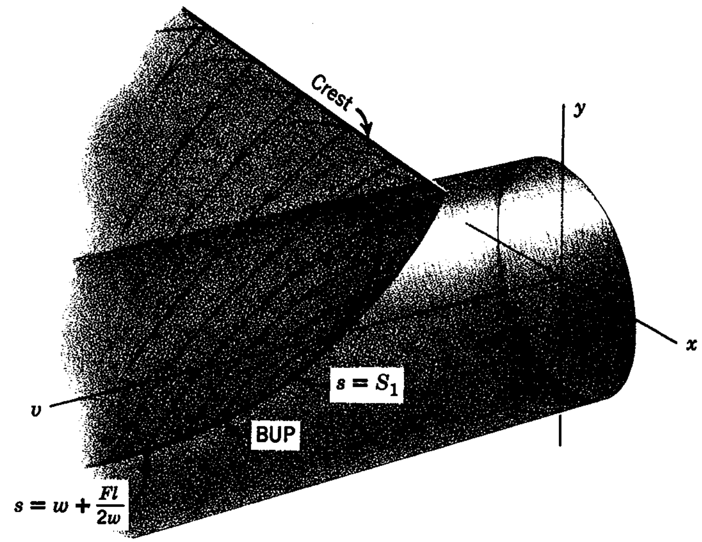


Figure 9.3.4

Using the barrier equations (9.3.3) and forming the 3 × 2 matrix

$$\frac{\partial(x, y, v)}{\partial(s, t)}$$

we can investigate the possibility of its rank falling below 2. Such happens when

$$\tau = \tau_T = \frac{w + \sqrt{2s^2 - w^2 - 2Fl}}{F}. \qquad (9.3.10)$$

Observe that for $s \geqslant S$, the argument of the radical is positive and that

$$\tau_0 \leqslant \frac{w}{F} < \tau_T$$

so that the presence of τ_T cannot interfere with the formation of the crest.

Does the barrier actually terminate when $\tau = \tau_T$? If so, in kinematic terms, what is the reason?

Problem 9.3.1. The Monotropic Rocket. Investigate the one-player game which differs from the present one only in that the thrust vector (of length F) always points directly forward. Thus P must move in a straight line and he has no volition.

The BUP will be the same as before, but now the barrier paths will fail when $s < S_1$, where S_1 is a root of

$$s^3 - w^2 s - wFl.$$

But nevertheless the natural barrier is complete. (See Figure 9.3.4.) Obtain these results as part of the full solution. What is the kinematic significance of the paths that begin in the nonuseable part?

9.4. THE ISOTROPIC ROCKET: THE ENVELOPE BARRIER

To complete the solution of this game of kind we must apply the ideas of Part III of Section 8.5. We shall construct curves, $\mathscr{D}^+$ and $\mathscr{D}^-$ in the nonuseable part of $\mathscr{C}$. Passing a semipermeable surface through them, each curve will be the envelope of constituent optimal paths.

In this section, as most of results will be *on* $\mathscr{C}$, we prefer new labels for its parametrization. We shall write

$$x = l \sin \theta$$
$$y = l \cos \theta$$

and the third coordinate shall be simply v. Thus we have replaced s_1 by θ and s_2 by v. We shall use v, θ as coordinates on $\mathscr{C}$.

As we did in Section 9.3, we can write on $\mathscr{C}$

$$l\dot{r} = x\dot{x} + y\dot{y}$$
$$= x\left(-F\frac{y}{v}\sin\phi + w\sin\psi\right) + y\left(F\frac{x}{v}\sin\phi - v + w\cos\psi\right)$$
$$= l[w\cos(\psi - \theta) - v\cos\theta].$$

Thus whenever

$$\frac{v\cos\theta}{w} \leqslant 1 \tag{9.4.1}$$

we can pick $\quad \psi = \breve{\psi} = \breve{\psi}(v, \theta) = \theta + \cos^{-1}(v\cos\theta/w) \quad$ (9.4.2)

so that $\dot{r}$ is 0 regardless of ϕ. Hence, an essential hypothesis of Theorem 8.5.1 is fulfilled.

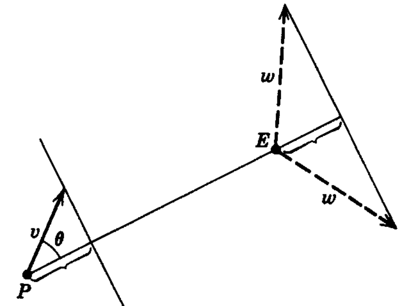


Figure 9.4.1

The condition (9.4.2) does not define $\breve{\psi}$ uniquely. There are two possibilities when $(v\cos\theta)/w < 1$. These are portrayed in Figure 9.4.1. The condition that the distance PE remain fixed as l is that the v and w vectors have the same projection (marked by braces) on the line PE. The choice between the dashed vectors is at E's disposal. Because evasion is his objective, we presume that he selects the one on the opposite side of the line PE from the v vector. A brief study of Figure 9.3.1a shows that this choice renders $\sin(\breve{\psi} - \theta) \geqslant 0$.

Assuming $\psi = \breve{\psi}$, we write, in the language of Theorem 8.5.1 the KE of the game G_1. First we have, from the KE of G,

$$\dot{x} = l\cos\theta\dot{\theta} = -F\frac{l}{v}\cos\theta\sin\phi + w\sin\breve{\psi}$$
$$\dot{y} = -l\sin\theta\dot{\theta} = F\frac{l\sin\theta}{v}\sin\phi - v + w\cos\breve{\psi}.$$

Multiplying them by $\cos\theta$ and $-\sin\theta$ and then adding leads to

$$l\dot{\theta} = -\frac{Fl}{v}\sin\phi + v\sin\theta + w\sin(\breve{\psi} - \theta).$$

We can now write the KE of G_1:

$$\dot{\theta} = -\frac{F}{v}\sin\phi + \frac{v\sin\theta + \sqrt{w^2 - v^2\cos^2\theta}}{l} \tag{9.4.3}$$
$$\dot{v} = F\cos\phi.$$

Putting

$$Z = v\sin\theta + \sqrt{w^2 - v^2\cos^2\theta}$$

the retrograde form is

$$\mathring{\theta} = \frac{F}{v}\sin\phi - \frac{Z}{l} \tag{9.4.4}$$
$$\mathring{v} = -F\cos\phi.$$

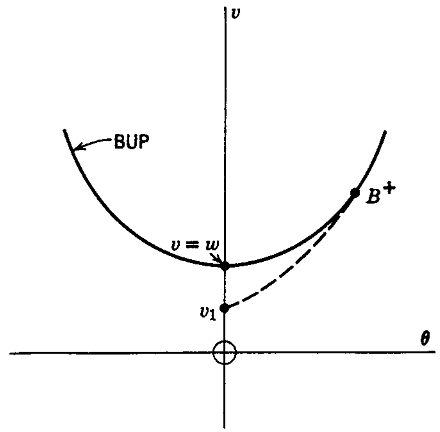


Figure 9.4.2

In the (θ, v)-plane, the BUP has the equation

$$v = \frac{w}{\cos\theta}$$

and B^+ has the coordinates

$$v = S = \sqrt{w^2 + K^2}$$
$$\theta = \beta$$

where (recalling (9.3.5)) $K = \sqrt{Fl}$ and $\cos\beta = w/S$, $\sin\beta = K/S$.

Our immediate objective is to draw a semipermeable surface of the game G_1 (with KE (9.4.3)) through B^+, such as the dashed curve in Figure 9.4.2. It is to reach the v-axis at the $v = v_1 \geqslant 0$.

The equations (9.4.4) are of the form

$$\overset{\circ}{\theta} = a\sin\phi - c$$
$$\overset{\circ}{v} = -b\cos\phi$$

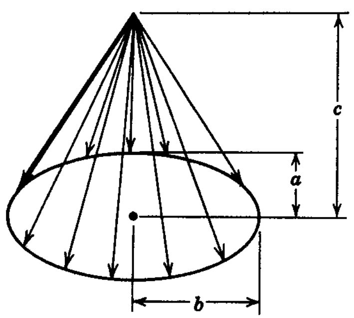


Figure 9.4.3

with a, b, $c > 0$. The vectogram, if $a \leqslant c$, appears as in Figure 9.4.3. For semipermeability, if we are working on the right side ($\theta > 0$), the heavy arrow must be chosen. If $a > c$, no surface locally exists.

The heavy arrow is the one that minimizes $\overset{\circ}{\theta}/\overset{\circ}{v}$ and so is determined

by

$$0 = \overset{\circ}{v}\frac{d}{d\phi}\overset{\circ}{\theta} - \overset{\circ}{\theta}\frac{d}{d\phi}\overset{\circ}{v} = -b\cos\phi(a\cos\phi) - (a\sin\phi - c)(b\sin\phi)$$
$$= -ab + bc\sin\phi$$

so that

$$\sin\bar{\phi}\ (\text{or } \bar{\bar{\phi}} \text{ in the language of the theorem}) = \frac{a}{c}$$

and so

$$\overset{\circ}{\theta} = -\frac{c^2 - a^2}{c}, \qquad \overset{\circ}{v} = -\frac{b}{c}\sqrt{c^2 - a^2}. \tag{9.4.5}$$

More conveniently,

$$\frac{dv}{d\theta} = \frac{\overset{\circ}{v}}{\overset{\circ}{\theta}} = \frac{b}{\sqrt{c^2 - a^2}} = \frac{F}{\sqrt{Z^2/l^2 - F^2/v^2}} = \frac{K^2}{Q} \tag{9.4.6}$$

where
$$Q = \sqrt{Z^2 - K^4/v^2}.$$

At B^+ we note that

$$Z = S\sin\beta + \sqrt{w^2 - S^2\cos^2\beta} = K$$

and

$$Q = \sqrt{K^2 - K^4/S^2} = \frac{K}{S}\sqrt{(w^2 + K^2) - K^2} = \frac{wK}{S}. \tag{9.4.7}$$

The immediate problem can be stated in terms of passing an integral of (9.4.6) through B^+. We note that at B^+, this curve will be tangent to the BUP. For the former has slope

$$K^2\Big/\frac{wK}{S} = \frac{SK}{w}$$

and the latter

$$\frac{d}{d\theta}\left(\frac{w}{\cos\theta}\right)_{\theta=\beta} = \frac{w}{\cos^2\beta}\sin\beta = w\left(\frac{S}{w}\right)^2\frac{K}{S} = \frac{SK}{w}.$$

To finish the barrier problem completely, we should prove three things:

(1) As we integrate the differential equation always we have $a \leqslant c$.
(2) The integral through B^+ reaches the v-axis at $v_1 \geqslant 0$.
(3) The paths in the original $\mathscr{E}$ with initial conditions along the integral curve constitute a new portion of the barrier which joins to the old without leaks, and together the two seal off a portion of $\mathscr{E}$.

We have not succeeded in proving these statements fully but what is unproven seems likely. We note in regard to them:

1. The condition $a \leqslant c$ is tantamount to $Q^2 \geqslant 0$. Now (9.4.7) shows Q is real at B^+ and therefore the integral curve can be extended for *some* positive distance from B^+ at least.
2. We have proved (omitted here) that if Q remains sufficiently large along the curve, then (2) will hold.
3. See Corollary 8.5.1 and the succeeding text.

Finally, we show that, if the escape condition $w^2 \geqslant 2Fl$ does not hold, then (1) and (2) cannot be true. For, when $\theta = 0$,

$$Z = \sqrt{w^2 - v^2}$$

and

$$Q^2 = w^2 - v^2 - \frac{K^4}{v^2}.$$

We are interested in the latter for $0 \leqslant v \leqslant w$. Near both ends of this range $Q < 0$. Its max occurs when

$$\frac{dQ^2}{dv^2} = -1 + \frac{K^4}{v^4} = 0 \qquad \text{or when} \quad v = K$$

and here

$$Q^2 = w^2 - K^2 - K^2 = w^2 - 2Fl.$$

Thus when the escape condition fails, Q^2 cannot be positive anywhere on the segment from (0, 0) to (0, w) and the integral curve from B^+ must terminate before reaching there.

But, with capture prevailing, a terminating, nonbounding barrier is just what we should expect!

What does all this mean in terms of the kinematics of the original problem? A crude view of things is shown in Figure 9.4.4. First, when the outcome is neutral, P pursues E. The motion of each is elementary; E travels straight and P with a constant direction acceleration. Ultimately E reaches the capture circle (X_1); the motions of both have been such that the (relative) contact is tangential. Now E, by playing $\psi = \breve{\psi}$ (or by picking the direction shown in Figure 9.4.1) stays on the circle. During this phase P picks a ϕ yielding the heavy arrow in Figure 9.4.3. Now P and E in the realistic space are traversing paths of a fairly intricate type. Finally, when E arrives at some point X_2 on the capture circle he will be

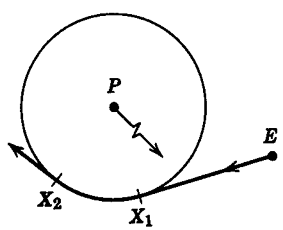


Figure 9.4.4

able to leave it with no danger of immediate capture. Thus he has completed a maneuver which might be called "sidestepping with contact."

Exercise 9.4.1. Complete Figure 9.3.2 by drawing in the new envelope barrier as best you conceive it.

9.5. TWO REMARKABLY DISSIMILAR GAMES IN THE SAME SETTING

The two points P and E have each simple motion in a half-plane bounded by a line $\mathscr{L}$. Their speeds are to be arbitrary; as only the ratio matters, we shall let P have speed unity and E, speed $w (w \gtreqless 1)$. Capture, as usual, is to occur when the distance $PE < l$.

1. In the first, called the *lifeline game*, E's objective is to reach $\mathscr{L}$ prior to capture and, naturally, P's objective is the contrary.[6]

2. The distinction from the former game is that now reaching $\mathscr{L}$ will be fatal to E. That is, capture will be considered to have occurred if *either* $|PE| < l$ or if E crosses $\mathscr{L}$. We perceive a generalization of both the wall pursuit game (Example 6.4.1) and the interception of a straight flying evader (Example 8.6.1). The novelty here is that E is no longer confined to $\mathscr{L}$ but is free to roam one of the half-planes bounded by it. The name here will be the *deadline game*.

Despite the parallelism of their formulations, the solutions are quite diverse. The former has an artificial barrier; the latter embodies one of envelope type.

Clearly in the game 1, all is trivial if $w > 1$. For E is free to put as great a distance as he pleases between the two players and then to streak unhindered to $\mathscr{L}$: all of $\mathscr{E}$ is the escape zone. Thus, here we will always suppose $w \leqslant 1$.

On the other hand, game 2 is trivial if $w < 1$. For then E can always be caught, even if $\mathscr{L}$ be disregarded. The latter's presence can only make matters worse for E: all of $\mathscr{E}$ is the capture zone. We thus will suppose $w \geqslant 1$.

For both problems we choose the coordinates shown in Figure 9.5.1a. The common KE are

$$\dot{y}_1 = \cos \phi$$
$$\dot{y}_2 = w \cos \psi$$
$$\dot{x} = w \sin \psi - \sin \phi.$$

[6] This contest would be same as the simple blocking game (Example 6.8.2) if we took $w = 1$. Because here we are interested in the game of kind aspect only, and in our former treatment the payoff was continuous (the distance E can advance toward $\mathscr{L}$ before capture), the sidelines lose a great deal of their interest. Hence, due to the arbitrary speed ratio, we are essentially generalizing our earlier problem.

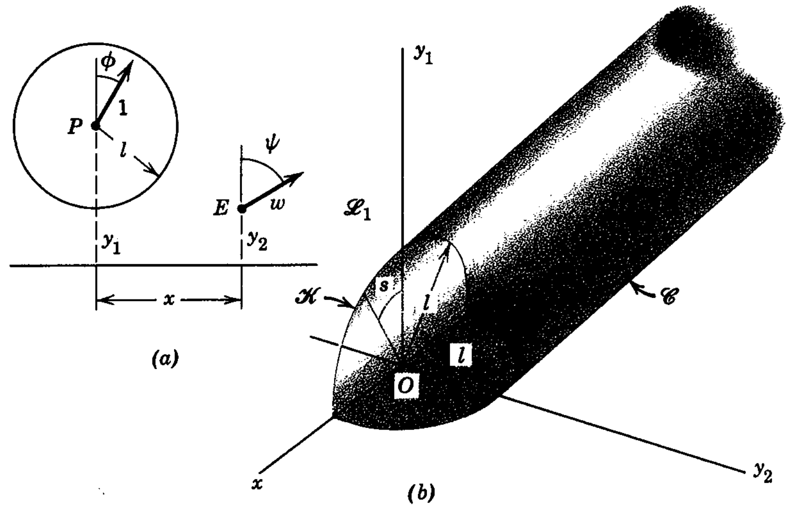


Figure 9.5.1

It is a familiar matter now for us to derive the ME_1:

$$\max_{\psi} \min_{\phi} [(\nu_1 \cos \phi - \nu_3 \sin \phi) + w(\nu_2 \cos \psi + \nu_3 \sin \psi)] = 0$$

so that, if

$$\rho_1 = \sqrt{\nu_1^2 + \nu_3^2}, \qquad \rho_2 = \sqrt{\nu_2^2 + \nu_3^2}$$

then

$$\cos \bar{\phi} = -\frac{\nu_1}{\rho_1}, \qquad \sin \bar{\phi} = \frac{\nu_3}{\rho_1}$$

$$\cos \bar{\psi} = \frac{\nu_2}{\rho_2}, \qquad \sin \bar{\psi} = \frac{\nu_3}{\rho_2}.$$

Thus the ME_2 is

$$-\rho_1 + w\rho_2 = 0$$

and the RPE

$$\mathring{y}_1 = \frac{\nu_1}{\rho_1}$$

$$\mathring{y}_2 = -w\frac{\nu_2}{\rho_2}, \qquad \mathring{\nu}_i = 0.$$

$$\mathring{x} = \nu_3\left(\frac{1}{\rho_1} - \frac{w}{\rho_2}\right)$$

The reduced space $\mathscr{E}$ appears in (b) of the figure. It is a quarter space ($y_1 \geqslant 0, y_2 \geqslant 0$) deprived of the interior of the cylinder $\mathscr{C}$. The latter corresponds to the points where $|PE| = l$; its axis lies on the 45° line of the y_1y_2-plane and it is such that its vertical and horizontal (but not axis-normal) sections are circles of radius l. The plane $y_2 = 0$, of course, corresponds to E's being on $\mathscr{L}$ and will be denoted by $\mathscr{L}_1$; $\mathscr{L}_1$ and $\mathscr{C}$ meet at the semicircle $\mathscr{K}$.

Example 9.5.1. The lifeline game. Thinking of **x** in $\mathscr{E}$, we envisage its meeting $\mathscr{L}_1$ as victory for E and $\mathscr{C}$ as victory for P. Thus, if both a capture and escape zone exist they must be separated by a semipermeable surface which separates $\mathscr{L}_1$ and $\mathscr{C}$. Hence it must pass through $\mathscr{K}$. Here is a pristine example of an artificial barrier.

Conversely, if we can pass such a surface through $\mathscr{K}$ which separates $\mathscr{E}$ such that $\mathscr{L}_1$ and $\mathscr{C}$ lie one each in the two components, it is clear that these components will be the escape and capture zones.

We parametrize $\mathscr{K}$ by

$$y_1 = l \cos s$$
$$y_2 = 0$$
$$x = l \sin s, \qquad -\frac{\pi}{2} \leqslant s \leqslant \frac{\pi}{2}.$$

A normal ν to it and on it must satisfy

$$\nu_1(-l \sin s) + \nu_3(l \cos s) = 0$$

so that we may take

$$\nu_1 = \cos s, \qquad \nu_3 = \sin s.$$

We invoke the ME_2 to obtain ν_2. As $\rho_1 = 1$, we have

$$\rho_2 = \frac{1}{w} = \sqrt{\nu_2^2 + \sin^2 s}$$

or

$$\nu_2 = \pm\sqrt{(1/w^2) - \sin^2 s}$$

and $\pm$ must be $+$ to enable the vector to point into $\mathscr{E}$.

As the state variables do not appear on the right in the KE, all $\mathring{\nu}_i = 0$, we can write the equations for the barrier at once, the general formula being x_i = (initial value of x_i) + (constant) $\times$ τ. We have here

$$y_1 = (l + \tau) \cos s$$
$$y_2 = w\tau\sqrt{1 - w^2 \sin^2 s} \qquad (9.5.1)$$
$$x = (l + (1 - w^2)\tau) \sin s.$$

To gain an idea of the nature of this ruled surface, let us extend the paths backward beyond $\mathscr{C}$. Putting $\tau = -l$:

$$y_1 = 0$$
$$y_2 = -wl\sqrt{1 - (w \sin s)^2} \qquad (9.5.2)$$
$$x = wl(w \sin s).$$

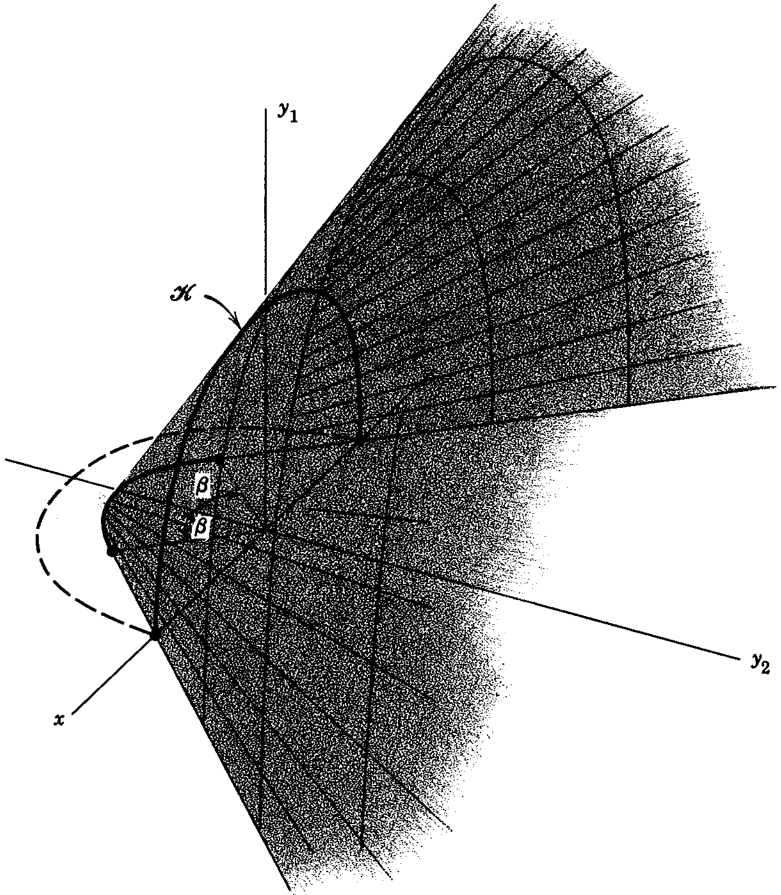


Figure 9.5.2

If we define the angle $u = u(s)$ by

$$\sin u = w \sin s$$

so that u ranges from $-\beta$ to $+\beta$, where β is the first quadrant angle with $\sin \beta = w$. Thus the curve (9.5.2) is the circular arc of radius $wl(\leqslant l)$ in the xy_2-plane,

$$y_1 = 0$$

$$y_2 = wl \cos u$$

$$x = wl \sin u, \qquad -\beta \leqslant u \leqslant \beta$$

and the surface (9.5.1) can be drawn by drawing straight lines between the corresponding points of it and $\mathscr{K}$. If $w < 1$, the result is as sketched in Figure 9.5.2, which is a sort of half horn through $\mathscr{K}$ and clearly fulfills its office of separating $\mathscr{L}_1$ and $\mathscr{C}$.

To get an idea of what's what in the realistic space, let us imagine the curve which is a horizontal section of this surface at height $y_1 > l$. In the realistic space this situation corresponds to P's being at a distance y_1 from $\mathscr{L}$. The section itself can be regarded as a picture of the realistic space: the capture circle will be the section cut of $\mathscr{C}$ and the section of $\mathscr{B}$ will appear as the other curve in Figure 9.5.3a. If E starts from any point below this curve he can reach $\mathscr{L}$ with impunity; from any point above P can catch him first.

If $w = 1$, the barrier surface of Figure 9.5.2, coincides with the upper half of the cylinder $\mathscr{C}$; the curve in Figure 9.5.3 coincides with the lower half of the capture circle (see (b)). What is the meaning?

The answer is a static barrier.

Such requires that $\dot{y}_1 = \dot{y}_2 = \dot{x} = 0$. From the KE, this can occur only if $\cos \phi = 0$, $\sin \phi = \pm 1$, $w = 1$, $\sin \psi = \pm 1$, $\cos \psi = 0$. We can then normalize so that $\rho_1 = \rho_2 = 1$ and we must have further

$$\nu_1 = 0, \quad \nu_2 = 0, \quad \nu_3 = \pm 1.$$

The ME then being

$$\max_{\psi} \min_{\phi} [-(\pm 1) \sin \phi + (\pm 1) \sin \psi] = 0$$

plainly it is satisfied by the above ϕ and ψ. Then in the reduced space, static semipermeable surfaces are planes normal to the x_3-axis. For our problem we utilize the halves of the two such planes which are tangent to

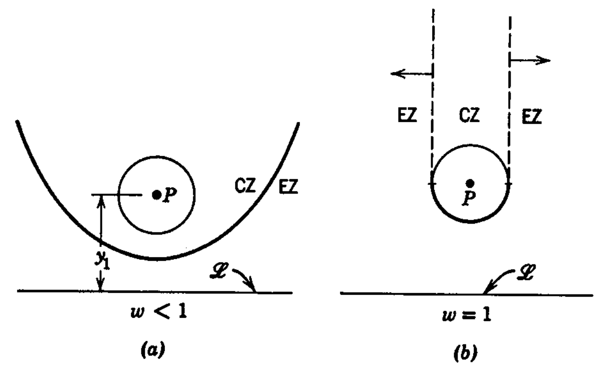


Figure 9.5.3

$\mathscr{C}$ and lie below. Such join into our former semicylinder to form a single smooth surface.

Returning to the realistic space with fixed y_1, we find we have adjoined to the barrier the two dashed half-lines of Figure 9.5.3b. The above $\pm$ takes a different sign on each; it is easy to see that the arrows drawn correspond to the flight directions of both players.

Note how this barrier agrees with our former analysis of the simple blocking game.

Problem 9.5.1. Show that (if $w < 1$) the same $\mathscr{B}$ can be reached by the geometric method. Consider the locus of points C such that

$$w(|PC| - l) = |EC|.$$

Show that the barrier corresponds to the set of positions such that this locus lies above $\mathscr{L}$ and is tangent to it. Explain why.

Research Problem 9.5.1. Solve the lifeline game when $\mathscr{L}$ is replaced by a circle. The playing space may exterior or interior to it. One might generalize further by taking $\mathscr{L}$ as an arbitrary curve.

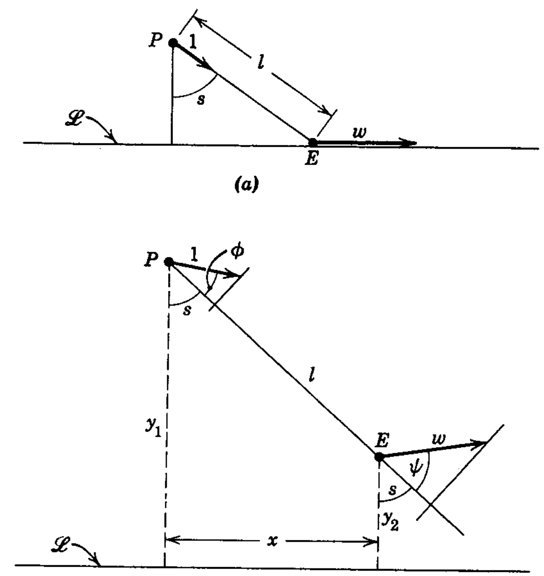
Example 9.5.2. The deadline game. In Figure 9.5.1b both $\mathscr{L}_1$ and $\mathscr{C}$ are now anathema to E. Their union can be regarded as the useable part.

Let us consider the semicircle $\mathscr{K}$. It corresponds to positions where E is on $\mathscr{L}$ and $|PE| = l$ such as in Figure 9.5.4a. Clearly E can break away only if his velocity component along PE can be made to exceed 1, that is, if $w \sin s \geqslant 1$. Defining β by $\sin \beta = 1/w$, $\cos \beta = \sqrt{w^2 - 1}/w$ this condition is also $s \geqslant \beta$. Now this s and that of Figure 9.5.1b are the same, and the only portion of $\mathscr{K}$ which might be considered as nonuseable is the arc of $\mathscr{K}$ called $\mathscr{K}_1$, where $-\beta \leqslant s \leqslant \beta$. Thus the two points of $\mathscr{K}$ with $s = \pm\beta$ are all that we can salvage of a BUP.

An envelope barrier seems a likely answer. To construct one we shall follow the scheme of Theorem 8.5.1.

We will redefine ϕ and ψ as shown in Figure 9.5.4b. To keep the distance PE fixed at l, E should choose $\breve{\psi}$ so that the projection of both velocity vectors on PE are equal. Of the two ways of doing this we shall, on intuitive grounds, suppose that he strives to make s increase; for example, in the figure he will endeavor to move counterclockwise around the capture circle. Then clearly

$$\begin{aligned} \cos \breve{\psi} &= \frac{1}{w} \cos \phi \\ \sin \breve{\psi} &= \frac{1}{w} \sqrt{w^2 - \cos^2 \phi} \end{aligned} \tag{9.5.3}$$



(b)

Figure 9.5.4

and for short we shall call the radical R. For coordinates on $\mathscr{C}$ we shall use s and y_2. As $x = l \sin s$ we see from the figure

$$\begin{aligned} (l \cos s)\dot{s} &= \dot{x} = w \sin(s + \breve{\psi}) - \sin(s + \phi) \\ &= \sin s \cos \phi + R \cos s - (\sin s \cos \phi + \cos s \sin \phi) \\ &= (R - \sin \phi) \cos s \end{aligned}$$

and also

$$\left.\begin{aligned} \dot{y}_2 &= -w \cos(s + \breve{\psi}) = R \sin s - \cos \phi \cos s. \\ l\dot{s} &= R - \sin \phi \end{aligned}\right\} \tag{9.5.4}$$

Adjoining

we have the KE of G_1. To obtain a semipermeable surface we proceed as with the isotropic rocket game:

$$0 = l\dot{s} \frac{d}{d\phi} \dot{y}_2 - \dot{y}_2 \frac{d}{d\phi} l\dot{s} = (R - \sin \phi)\left(\frac{\sin s \cos \phi \sin \phi}{R} + \cos s \sin \phi\right)$$
$$-(R \sin s - \cos s \cos \phi)\left(\frac{\cos \phi \sin \phi}{R} - \cos \phi\right)$$

which reduces to

$$w^2 \sin \phi \cos s + (w^2 - 1) \cos \phi \sin s - R \cos s = 0. \tag{9.5.5}$$

After further reduction (9.5.5) becomes a biquadratic equation in $\cos\phi$. For each root there is still a sign choice in evaluating $\sin\phi$. However, there are but two of these four possibilities that are solutions of (9.5.5). Such correspond to the extreme vectors of the vectogram. From the nature of our problem we wish $\dot\theta/\dot y_2$ negative, for during neutral play s clearly increases (to β) and y_2 decreases (to 0). Only one possibility achieves this. It turns out to be, and the reader may check it,

$$\cos\phi = \frac{w\cos s}{Q}$$
$$\sin\phi = \frac{1 - w\sin s}{Q} \qquad (9.5.6)$$
$$R = \frac{w(w - \sin s)}{Q}$$

where

$$Q = \sqrt{1 + w^2 - 2w\sin s}$$

and these values occasion, when substituted into (9.5.4),

$$\dot y_2 = -\frac{w(1 - w\sin s)}{Q}, \quad \text{which} \leqslant 0 \text{ when } \theta \leqslant \beta. \qquad (9.5.7)$$
$$\dot s = \frac{w^2 - 1}{lQ}.$$

The quotient yields

$$\frac{dy_2}{ds} = -\left(\frac{wl}{w^2 - 1}\right)(1 - w\sin s)$$

an equation which is to be integrated with the initial conditions: $s = \beta$,[7] $y_2 = 0$. We obtain

$$y_2 = \frac{l}{w^2 - 1} w(k - s - w\cos s) \qquad (9.5.8)$$

where $k = \beta + w\cos\beta = \beta + \sqrt{w^2 - 1}$. There is no reason why s should not parametrize $\mathscr{D}$, the envelope curve. The other two equations are easy to obtain. As $y_1 - y_2 = l\cos s$,

$$y_1 = \frac{l}{w^2 - 1}(wk - ws - \cos s)$$
$$y_2 = \frac{l}{w^2 - 1}(wk - ws - w^2\cos s) \qquad (9.5.9)$$
$$x = l\sin s$$

[7] We are working here on the side of $\mathscr{E}$, where $x > 0$.

which are the equations of $\mathscr{D}$, or rather the branch for which $x \geqslant 0$. Clearly the range of s in (9.5.9) is $[\beta, 0]$.

To obtain the semipermeable surface is easy; it is the union of $\mathscr{D}$ and its half-tangents properly oriented. The first equation is obtained, for example, from (9.5.9), if $y_1(s)$ means the latter's right side, by

$$y_1 = y_1(s) - y_1'(s)\tau.$$

The minus sign appears because s decreases (from β to 0) as we proceed retrogressively along $\mathscr{D}$. Of course, the τ appearing will not measure time of progress; this would be so only if s did along $\mathscr{D}$ and it plainly does not. But the distortion of τ does not change the surface, only its parametrization.

Thus the semipermeable surface finally is

$$y_1 = \frac{l}{w^2 - 1}[w(k - s) - \cos s + (w - \sin s)\tau]$$
$$y_2 = \frac{l}{w^2 - 1}[w(k - s) - w^2\cos s + w(1 - w\sin s)\tau] \qquad (9.5.10)$$
$$x = l[\sin s - \tau\cos s].$$

Because we are only working here on the $x \geqslant 0$ side, we break things off at the $x = 0$ and adjoin a symmetric image. Figure 9.5.5 shows the

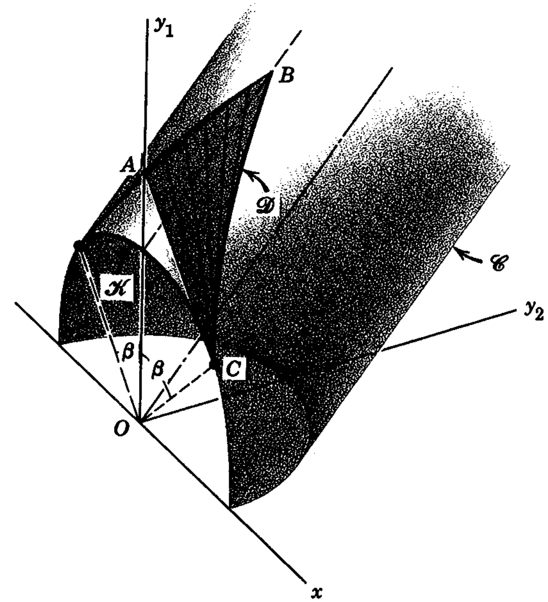


Figure 9.5.5

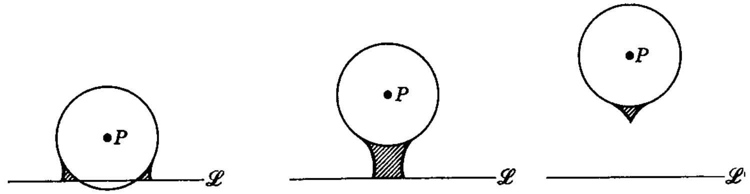


Figure 9.5.6

final envelope barrier. The arc CB is the envelope curve $\mathscr{D}$, with $s = \beta$ at C and 0 at B. As $\mathscr{D}$ and $\mathscr{K}$ are tangent at C, as has been shown, the tangent line CA to $\mathscr{D}$ lies in the x,y_1-plane. The remaining half-tangents to $\mathscr{D}$ are extended until they meet the $x = 0$ plane, which they do at the crest AB. This curve, as may be easily shown, rises (y_1 increases) on proceeding from A to B.

Let the letters A, B, C also designate the values of y_1 at the corresponding points. Then from simple computations

$$A(=y_1 \text{ at } A) = \frac{wl}{\sqrt{w^2-1}}$$

$$B = \frac{l}{w^2-1}[wk-1] = \frac{l}{w^2-1}\left[w\left(\sin^{-1}\frac{1}{w} + \sqrt{w^2-1}\right) - 1\right] \tag{9.5.11}$$

$$C = l\cos\beta = l\frac{\sqrt{w^2-1}}{w}.$$

To get an idea of what goes on in the realistic space, we consider, as before, sections of constant y_1, which can be regarded as pictures with P fixed and E variable. When $y_1 < C$ or $>A$ capture can be forced by P for no position of E; the capture zones for the remaining case are depicted in Figure 9.5.6 as shaded regions.

Exercise 9.5.1. What are the equations of the barrier sections in Figure 9.5.6?

Exercise 9.5.2. Investigate the limiting case as $w \to 1$. Show that when $w = 1$, the barriers are static. Their sections in Figure 9.5.6 would appear as inversions of those of the lifeline game: the configuration is the *upper* half of the capture circle plus half-tangents extending *downward* to $\mathscr{L}$.

Research Problem 9.5.1. Investigate the paths of P and E in the realistic space corresponding to $\mathbf{x}$ being on $\mathscr{D}$. Is there any simple interpretation to the curved path of P?

Research Problem 9.5.2. Generalize by allowing $\mathscr{L}$ to be the arc of a circle.

9.6. EXTENSIONS AND APPLICATIONS OF THE DEADLINE GAME

To start we take up (always $w > 1$)

Example 9.6.1. The one-sided deadline game. We envisage E as starting from, say, far to the left of P who is a distance $> l$ above $\mathscr{L}$. The object of E is to move to the right and pass between P and $\mathscr{L}$ without being captured. Of course, P's object is the contrary.[8]

The distinction between this problem and that of the previous section is that now symmetry is abandoned. The same barrier suffices here, but we no longer curtail it when $x = 0$. That is, the equations (9.5.10) still supply the relevant semipermeable surface, but the parameter range is to be enlarged to $s \leqslant \beta, \tau \geqslant 0$. In the counterpart of Figure 9.5.6, we retain only the right boundary of the shaded region, producing it if necessary until it meets $\mathscr{L}$.

But here we encounter a novelty. Our tenet that the barrier separate $\mathscr{E}$ is violated. However, validity has not absconded but changed its nature. Topology has intruded. Before expounding further, it is well to look at our now asymmetric barrier.

Consider the curve labeled $\mathscr{D}$ in Figure 9.5.5. If it is produced beyond B it will, as its equations show, spiral around the cylinder $\mathscr{C}$. The barrier then, the union of the half-tangents, will be a sort of flared helicoid. All these rays extend upward, for we see from (9.5.10) that $\partial y_1/\partial\tau > 0$. All then will meet a horizontal section provided they emanate from a point of $\mathscr{D}$ lying below its plane. Thus a high such section will consist of a spiral whose number of turns is greater with increased height. Some cases with progressively greater y_1 are shown in Figure 9.6.1.

The situation fits a game whose payoff is a non-negative integer.

Example 9.6.2. Looping the blockader. Let E start, say, from the left, as in the last example, and finish on the far right. The payoff is the number of counterclockwise revolutions he can make about P without being captured or touching $\mathscr{L}$, or it is -1 if E cannot even pass without being captured.

It is easy to apprehend the solution. The curves of Figure 9.6.1 (for a fixed position of P, of course) separate starting points of different Value in much the same topological way as the sheets of a Riemann surface

[8] In terms of football, we can think of a ball carrier trying to pass safely between a sideline and a slower, opposing tackler posted near it.

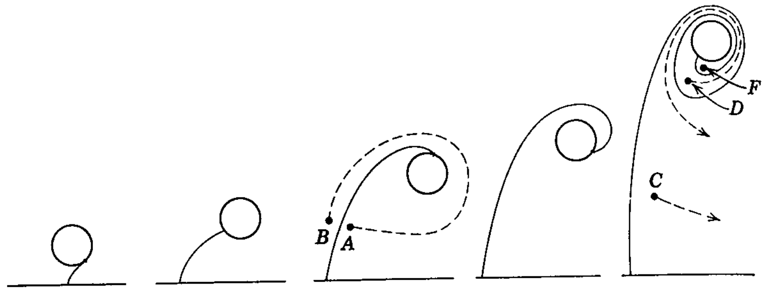


Figure 9.6.1

separate the branches of a function.[9] For example, if E starts from C, D, F, the Value is 0, 1, 2, respectively.

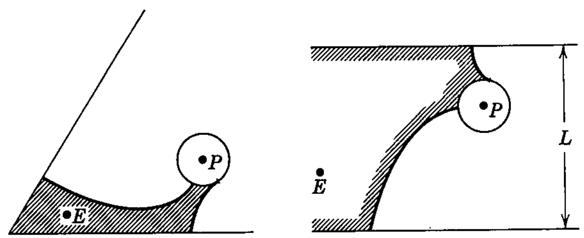
Example 9.6.3. The cornered rat game. The rat E is entrapped in a corner (of any angle) by the cat P.[10] When is escape possible?

In certain circumstances—one is shown in Figure 9.6.2a—by drawing the barrier sections of the type just discussed to each wall we can delineate the capture zone. Here E cannot escape if his starting point lies within the shaded region.

Further analysis of this problem will be presumed similar to that of the one which follows and so will not be discussed separately.

Example 9.6.4. Patrolling a channel. Let E be confined between two parallel lines of distance apart L. His objective is to get past P without being captured.

Figure 9.6.2b is an instance in which E seeks to escape to the far right. Again two barrier sections mark off a capture zone (shaded) of starting



(a) (b)

Figure 9.6.2

[9] For example, note the Value changes from 0 to -1 in going from A to B via the dashed line in the figure.

[10] Speeds, motions, and capture are as in the deadline game and the same is true of the ensuing examples.

points from which E's passage can be prevented through proper play by P.

In both these problems it is intuitively clear that for sufficiently favorable values of the parameters E should be able to pass P without capture from all (or possibly all but some insignificant subset of) starting points.

Let us clarify a notion of intersecting barriers for sharper reasoning. Suppose, in the analysis of some general game of kind, we encounter two candidates for barriers which intersect as shown in Figure 9.6.3, where $n = 2$ for simplicity. Observe that they are oriented oppositely from our earlier cases such as the homicidal chauffeur game. There we saw that we were justified in erasing the barriers beyond their intersection; here we shall show that we should discard the entire barrier.

We suppose the barriers not static, that is, under optimal play $\mathbf{x}$ traverses them toward $\mathscr{C}$. Let $\mathbf{x}$ start from X_1, which is in the capture zone but very close to one barrier $\mathscr{B}_1$. Both players' adoption of the optimal (very near the neutral) strategy will occasion a motion nearly parallel to $\mathscr{B}_1$. Ultimately $\mathbf{x}$ will reach X_2 near the intersection. If the same strategies persevere, $\mathbf{x}$ will cross $\mathscr{B}_2$ and E will escape.

Note that P cannot prevent this escape. The intersection of the barriers does not entail constructive guessing by a player of which strategy—that relevant to $\mathscr{B}_1$ or $\mathscr{B}_2$—his opponent will use; an instantaneous mixed strategy is futile here. Let us suppose that at X_2, E decides on the strategy pertaining to $\mathscr{B}_1$. If P uses this, too, we have seen that $\mathbf{x}$ crosses $\mathscr{B}_2$ and E escapes. If P does anything else, E's strategy will occasion $\mathbf{x}$'s crossing of $\mathscr{B}_1$ and again he escapes. Thus *all* the nearby region belongs to the escape zone.

To see that this kind of intersection does occur in our game when L is sufficiently large, refer to Figure 9.6.4. To take into account the added side of the corridor we adjoin, in the obvious way, the new coordinates

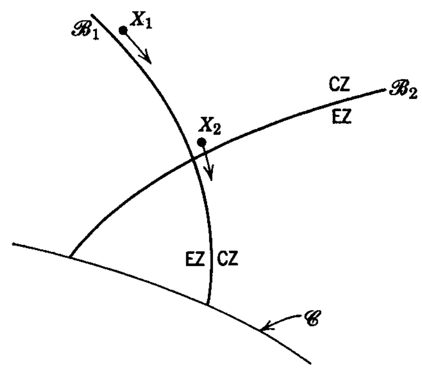


Figure 9.6.3

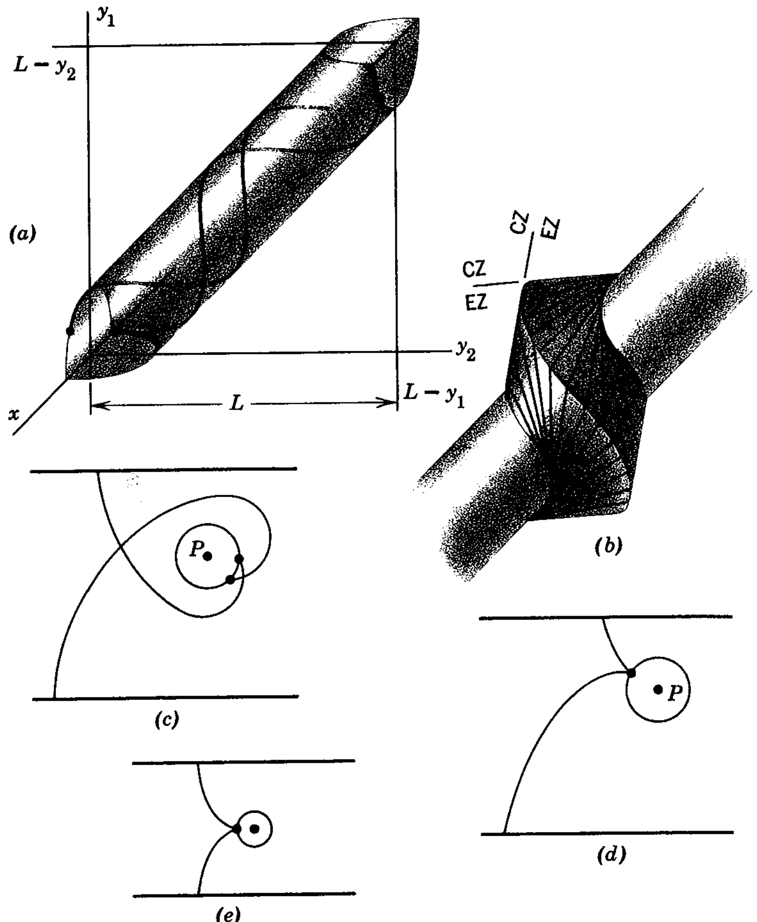


Figure 9.6.4

$L - y_2$ and $L - y_1$. The two $\mathscr{D}$ curves are intertwining spirals as shown at (a). At (b) we have a vivid attempt at depicting part of the intersecting barriers; the reader can convince himself that the orientation is as in Figure 9.6.3. Thus we conclude that whenever the channel is so wide that the two barriers intersect, as at (c), then E can pass from *any* starting position. We will show below existence (and find it) of a critical channel width where this kind of intersection begins, *regardless* of P's position in the channel.

Let L_c denote the critical width at which intersection commences if P is centered in the corridor ($y_1 = \frac{1}{2}L_c$). The sectional barriers must both

meet the capture circle with $s = \pi/2$ as shown at (e). Giving y_1 and s these values and $\tau = 0$ in (9.5.10), we ascertain that

$$L_c = \frac{2l}{w^2 - 1} w\left(k - \frac{\pi}{2}\right)$$

or

$$\boxed{L_c = \frac{2lw}{w^2 - 1}\left(\sqrt{w^2 - 1} - \cos^{-1}\frac{1}{w}\right)} \qquad (9.6.1)$$

The assertion we are after is

If $L > L_c$, E can pass from any starting position (P cannot guard the corridor) but such is not true if $L < L_c$. (9.6.2)

We need but show that $L > L_c$ is the condition that the two barrier sections intersect. We have seen that this is true when P is centered in the corridor. To prove it for arbitrarily located P, as at (d), it suffices to show that, when $L = L_c$, the two barrier sections meet at a point of the capture circle wherever P is. Such is tantamount to showing that the two spirals of Figure 9.6.4a coincide when $L = L_c$.

The equation of one spiral is given by (9.5.9); those of the other by replacing y_i by $L - y_i$ ($i = 1, 2$) and s by $\pi - s$. Our result follows by noting that when $L = L_c$, the final replacement follows automatically. For example, from (9.5.9) and (9.6.1),

$$L_c - y_1 = \frac{l}{w^2 - 1}\left\{2w\left(k - \frac{\pi}{2}\right) - [wk - ws - \cos s]\right\}$$

$$= \frac{l}{w^2 - 1}\{wk - w(\pi - s) - \cos(\pi - s)\}.$$

Thus the assertion (9.6.2) is proved.

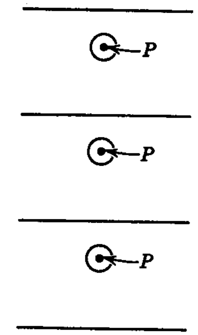


Figure 9.6.5

Example 9.6.5. The patrol line. Given a row of equispaced pursuers, under what conditions can a single evader E pass through the row without being captured?

By imagining corridor sides placed between each pair of adjacent P as shown in Figure 9.6.5, it would appear as though, by the principle of reflection, this game reverts back to the previous one. Thus a *sufficient* condition that the patrol line be effective is that the spacing $< L_c$. (9.6.3)

Research Problem 9.6.6. The Patrol Circle. Given a set of P equispaced around a circle, when can they effectively prevent the escape of a single E who is initially within the circle?

This problem bears the same relation to the cornered rat game as the above does to the corridor game.

Problem 9.6.1. Treat the limiting cases of the above games when $w = 1$. Show that in the cornered rat and corridor games, a capture zone always exists.

9.7. FURTHER GAMES

The examples in this section have not been fully solved. But they are of interest and possibly of importance. Because no more than suggestions are given for their treatment, the reader can regard them as research problems.

Bounded pursuit games

Here P and E move in the plane, both with simple motion, with E having the greater (or possibly equal speed). But E (and possibly also P) is confined to a subregion $\mathscr{R}$ bounded by one or more curves. This class of problem was suggested to us by the late J. von Neumann.

The deadline, cornered rat, and corridor[11] games are all instances. Can we handle other regions?

Suppose $\mathscr{R}$ is the interior of a circle. If it is of large radius, it is reasonable to suppose that results of the type depicted in Figure 9.5.6 will hold when P is near $\mathscr{L}$, now dished upward. Do the shaded regions retain their validity as capture zones? Certainly when E's starting point is within them he will be captured. But what of unshaded starting points? If it is true that the shaded region of the figure is the only capture zone and it changes continuously with P's motion, then E will be able to escape perpetually. For if not there would occur an instant when E would be able to cross the boundary of the zone; but he cannot be forced to do so because the boundary is semipermeable.

If $\mathscr{R}$ is the interior of a polygon, we might similarly endeavor to apply the cornered rat game to positions where P is near a vertex or the deadline game when P is near the central part of a side.[12]

We hesitate to stress these concepts overmuch, for it seems likely that there will be significant cases in which all of $\mathscr{E}$ is the capture or escape zone and barriers will not be the appropriate tool.

[11] Not exactly, for here E is not interested in passing P and can always escape. But if the corridor is cut off by a cross boundary at one point (semi-infinite strip) or at two (long, narrow rectangle), then from some starting positions E must pass P in order to survive and our previous ideas are applicable.

[12] Especially the former. For if $\mathscr{R}$ can be contained in a half-plane, then a non-vacuous capture zone will exist. For this zone of the deadline game will *a fortiori* hold for a more restricted region then the containing half-plane.

Connectivity must have a bearing (assuming P is similarly confined to $\mathscr{R}$). For in some cases P can clearly never catch E if he has to chase him around a closed circuit.

Dogfight games

Let P and E move in, say, the plane, each with such kinematics that their directions of motion are state variables (Example: bounded curvature). Let each have a capture region which extends forward (in the direction of motion) of his position. Figure 9.1.4 is a typical example. Each player strives to get his opponent within his own region prior to his being himself so caught. Thus we exemplify conflict between two single-seater airplanes, each armed with unturnable guns that point straight forward only.

Of course, a partie may be a draw in that neither player may be able to force the other into his capture region. To simplify the discussion we shall suppose this possibility excluded.

In the reduced space, $\mathscr{C}$ will consist of two surfaces, $\mathscr{C}_1$ and $\mathscr{C}_2$, which correspond to the boundaries of the two capture regions in the realistic space. As play proceeds $\mathbf{x}$ moves about in $\mathscr{E}$ until it first penetrates one of the $\mathscr{C}_i$. Which one determines the outcome.

We see the technique of solution at once. The two $\mathscr{C}_i$ will intersect, for both players being within their opponent's capture region, either one only, or neither, must all be admissible positions in any reasonable simulation of reality. Let the intersection consist of one (or more) curves. We pass a properly oriented semipermeable surface through these curves. It should separate $\mathscr{E}$ into two parts which are the winning starting positions of the two players.

Naturally dogfights may be formulated in other ways than a pure game of kind. For example, victory may require that one's opponent remain in one's capture region for some stipulated duration. Or a continuous payoff could be adopted which is an integral of the time of the vulnerability of one player less that of the other.

Battles of extermination

The players commence with certain numbers of "men"[13] and the kinematic equations describe the mode whereby these numbers are altered during play. The player loses the game who first loses all his "men."

Thus the contest is a game of kind and a barrier should delineate the starting positions from which one or the other player wins. We have seen how to construct such in discrete cases in Chapter 3. To adopt the same principle to continuous models, in the light of this chapter, is not hard.

[13] Soldiers or other munitions in a battle, checkers on a draughts board, etc.

9.8. APPLICATION TO STABILITY AND CONTROL

Such problems are one-player games with our customary KE:

$$\dot{x}_i = f_i(\mathbf{x}, \phi), \qquad i = 1, \ldots, n$$

and subject to the supplementary condition

$$f(0, 0) = 0.^{14} \qquad (9.8.1)$$

Here **x** is regarded as the "error" of some possibly unstable mechanical or electrical system and the ϕ_i are control variables whose function it is to obviate undesirable digressions from the central equilibrium state. This state is taken at $\mathbf{x} = 0$, which is the interpretation of (9.8.1). The designer has the option of making ϕ a function of **x**—just our conception of a strategy.

Various criteria for stability are extant, generally based on the asymptotic or ultimate behavior of the differential equations arising when $\phi = \phi(\mathbf{x})$ in the KE.

It seems reasonable that the theory of games of kind can offer an alternative approach. The boundary to all possible deviations from equilibrium will be a barrier and our techniques should be able to ascertain it. Indeed the idea is used in the concept of "controllability."[15]

[14] See, for example, La Salle's survey article [7] and the example in the Appendix.
[15] See References [5].

CHAPTER 10

Equivocal Surfaces and the Homicidal Chauffeur Game

10.1. INTRODUCTION

The remarkably eclectic and instructive game of the homicidal chauffeur has appeared fragmentarily in several earlier sections. To weld the bits into the full solution, we must introduce, in addition, the rich variety already embodied, a new type of singular surface. It has, of course, wider interest than for this one problem.

It is termed an *equivocal surface*, for one of the players has the choice of two distinct optimal strategies at each of its points. The phenomenon cannot exist in a one-player game; there is no counterpart in the calculus of variations. We will expound its theory in Section 10.5 by a detailed discussion of an example typical enough to embody all salient features.

For the homicidal chauffeur, both the games of kind and degree are interesting. We solve the former here by a purely geometric construction and treat the latter by methods partially so. This more transparent approach will clarify, confirm, and complete certain earlier analyses.

10.2. THE HOMICIDAL CHAUFFEUR: GEOMETRIC SOLUTION OF THE GAME OF KIND

We will exploit the symmetry of the game by speaking almost throughout as if E were on or to the right of P's line of travel. In other words, we will work in the right half of the plane in which lies $\mathscr{E}$, the reduced space.

To solve the game of kind we must construct the barrier. We have done so by our standard technique in Section 9.1, but in the one immediately following we shall obtain the barrier geometrically.

The game of degree, with time of capture as the payoff, hinges strongly on the barrier. As we have seen and will see again, it consists of two arcs of involutes of the circles we have termed $\mathscr{K}_+$ and $\mathscr{K}_-$.[1] When these arcs intersect, P cannot compel capture unless **x** is in the curvilinear triangle bounded by them and $\mathscr{C}$ (see Figure 9.1.2). But such a position, as we noted before, is a setup: E is planted directly in front of the oncoming vehicle. Thus we are justified in regarding the intersecting barriers as the general criterion of escape.

Of course, we can solve the game with capture time in this case (the solution exists only in the triangle), but it is of much the greater moment to work with nonmeeting barriers. These curves still play an important role. They delineate the positions in which optimal play leads to a straightforward chase from those where a swerve[2] maneuver occurs.

We recall that E travels in the plane with simple motion of speed w_2. Thus his vectogram in the reduced space $\mathscr{E}$, that is, in a plane rigidly attached to P's vehicle, is evident: we still have a circular vectogram of radius w_2.

But P's vectogram (the ϕ-vectogram) is not so apparent. Our first step—Lemma 10.2.1 below—is to describe it in geometric terms. Recall that P travels at the fixed speed w_1 ($> w_2$) and with his radius of curvature bounded absolutely by a given R. He navigates by choosing his curvature ϕ/R (with $-1 \leqslant \phi \leqslant 1$) at each instant.

LEMMA 10.2.1. At any point X of $\mathscr{E}$, the ϕ-vectogram is constructed by the steps (see Figure 10.2.1a):

1. From X draw a downward vector XA of length w_1.
2. Through A draw a line H perpendicular to OX (O = origin). H contains the headline of the vectogram.
3. From X draw the vectors $XU_\pm$, terminating on H, which are respectively perpendicular to the lines from $(O, \pm R)$ to X. These vectors are the extreme members of the vectogram, $XU_\pm$ corresponding to $\phi = \pm 1$.

Proof. Put $r = R/\phi$ $(-1 \leqslant \phi \leqslant 1)$, so that the center of the curvature selected by P is $C = (r, 0)$ as in (b) of the figure. Put $d = |CX|$.

Now P's pivoting about C in the realistic space is equivalent to X's pivoting about C in $\mathscr{E}$ in the opposite direction but with the same angular speed. The resulting velocity of X will be XU perpendicular to CX and of magnitude $w_1 d/r$.

Now consider the triangles OCX and AXU. They have two pair of mutually perpendicular sides, and the length ratios of these sides is w_1/r in both cases. Thus they are similar, and the third sides OX and AU are also

[1] They have centers $(\pm R, 0)$ and radius Rw_2/w_1.
[2] See Section 1.5.

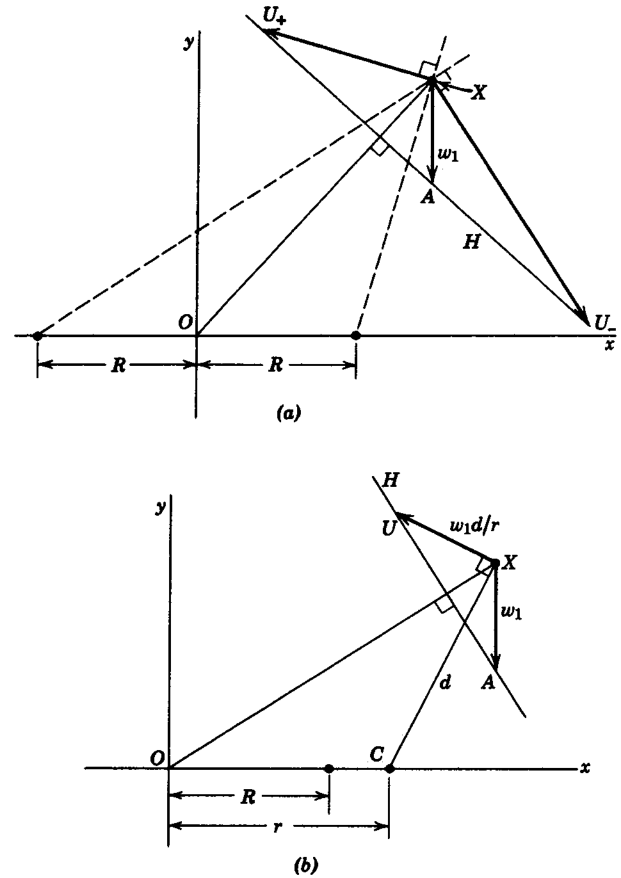


Figure 10.2.1

perpendicular. This proves the contention in 2 and 3 follows because of the bounds on ϕ.

LEMMA 10.2.2. The semipermeable direction (properly oriented for the barrier) at X is constructed by first drawing a circle of center U_+ and radius w_2. The sought direction is that of the lower[3] tangent (XD in Figure 10.2.2a) from X to this circle. The local optimal strategies are: $\bar{\phi} = 1$ and $\bar{\psi}$ is such that E's velocity is U_+D, where D is the point of tangency.

[3] "Lower" applies to X in the upper half-plane as sketched. For other X, the choice of tangent is fixed by continuity.

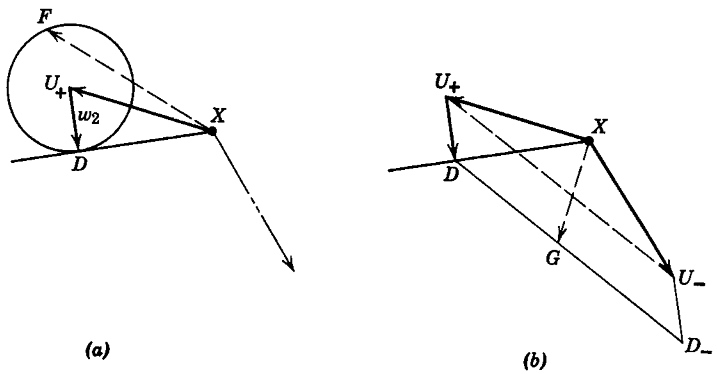


Figure 10.2.2

Proof. If P plays XU_+ ($\phi = 1$) it is clear that the resultant velocities available to E are those with vectors extending from X to points of the circle (such as XF in the figure). None penetrates XD in the downward direction.

If E plays U_+D, let U_-D_- be the translate of E's velocity drawn from U_- (see (b) of the figure). The resultant velocities available to X are those with vectors XG, where G is a point of the closed segment DD_-. None penetrates XD upwardly.

LEMMA 10.2.3. The semipermeable surfaces (in the right half-plane and oriented as above) are involutes of the circle $\mathscr{K}_+$.

Proof. From X draw the tangent XJ shown in Figure 10.2.3 to $\mathscr{K}_+$.

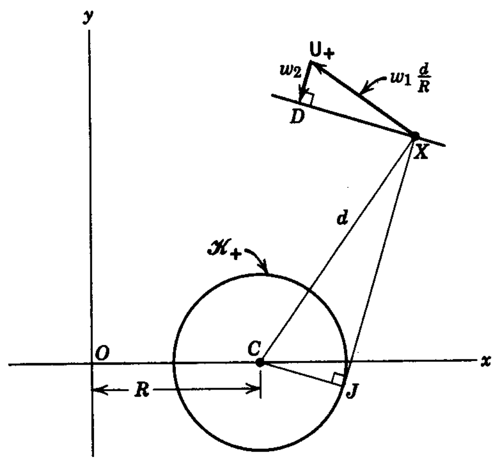


Figure 10.2.3

In the right triangle CJX, the ratio (short leg: hypotenuse) is w_2R/w_1d where $d = |CX|$. The proof of Lemma 10.2.1 shows that $|XU_+| = w_1d/R$ so that this ratio is the same for the right triangle U_+DX. Thus the triangles are similar. As, from Lemma 10.2.1, XU_+ is perpendicular to CX, it follows that semipermeable direction XD is perpendicular to XJ. Classical knowledge of direction fields and differential equations now establishes the lemma.

LEMMA 10.2.4. Draw the lower ray from O, which is tangent to $\mathscr{K}_+$. The involute of Lemma 10.2.3 ceases to be semipermeable below this ray.

Proof. Referring to Figure 10.2.2b, it is clear that the semipermeable property of XD fails should the segment DD_- lie on the same side of XD as

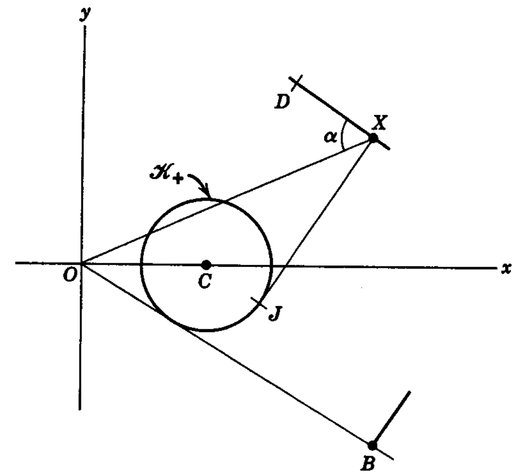


Figure 10.2.4

U_+. Thus it suffices to show that, as X moves clockwise along the involute, the angle D_-DX decreases and it becomes zero when X meets the above ray.

Now DD_- is parallel to the headline U_+U_- and we recall that the latter is perpendicular to OX. Thus we need show that, as the involute unwinds, the complementary angle α, between XD and OX (see Figure 10.2.4), increases and becomes a right angle when X reaches the ray (such as at B in the figure). But this is obviously true, as α steadily approaches a right angle as OX and JX approach coincidence; which they attain when X is at B.

We are now in a position to (re)state the construction of the barrier. We can suppose a capture region bounded by an arbitrary convex curve $\mathscr{C}$ about O.

THEOREM 10.2.1. The right barrier is constructed as follows, the left one being treated symmetrically:

Draw the involute, unwinding clockwise, of $\mathscr{K}_+$ which contacts $\mathscr{C}$ and is otherwise exterior to it. If this is impossible, there is no barrier. Otherwise the barrier is the arc of this involute extending (in the unwinding sense) from the contact point with $\mathscr{C}$ until the first point where it either meets the left barrier or the lower ray through O tangent to $\mathscr{K}_+$.

The only detail of the proof not supplied by the preceding lemmas is that the useable part of $\mathscr{C}$ is spanned by the contact points of the involutes. However, if we assume $\mathscr{C}$ is smooth[4] and accept as known that the useable part is a connected arc on the upper part of it, then, as tangency to the barrier is equivalent to the BUP condition (Section 8.5.I), all follows.

10.3. THE PRIMARY SOLUTION OF THE HOMICIDAL CHAUFFEUR GAME OF DEGREE

The term *primary* applies to the integration of the RPE with initial conditions on $\mathscr{C}$ to obtain the paths terminating thereon. In most examples it is the first integration we do and an important one; in those with no inhibiting singular surfaces indeed it yields the full solution.

The present problem is exceptional in that the primary integration is of minor significance. The optimal paths on the right are shown in Figure

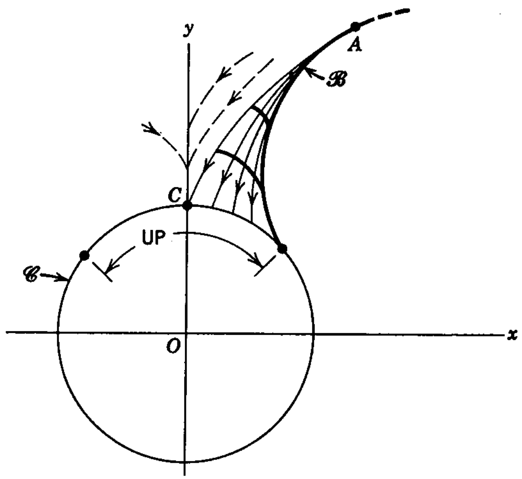


Figure 10.3.1

[4] For general $\mathscr{C}$ (which the barrier may meet at a corner) it is not hard to discuss the game of kind in terms of embedding family of semipermeable involutes, somewhat in the manner of Section 8.5. II and not explicitly invoke useable parts at all.

10.3.1 as well as the curves of constant V. For simplicity, we suppose the $\mathscr{C}$ to be a circle centered at O.

We know that when the initial distance between P and E is appreciably large, the chase will always culminate with **x** on the universal surface on the y-axis. In the realistic space we know the interpretation: P follows E along a straight path until capture occurs when E meets the foremost point (C in the figure) of $\mathscr{C}$.

The primary paths mark the exception: For E initially close and not too much off center, P captures with a sharp turn only. The path of E is straight; its optimal direction is such that he enters $\mathscr{C}$ perpendicularly in the sense of the relative motion of both players.

It is not hard to show that the envelope principle[5] holds: the barrier $\mathscr{B}$ is tangent to the curves of constant V, and $\mathscr{B}$ itself is an optimal path of minimax time of capture.

Problem 10.3.1. We have stated that the optimal paths enter the useable part perpendicularly to $\mathscr{C}$. Yet the barrier, similarly an optimal path, meets $\mathscr{C}$ tangentially. In view of continuity of the optimal strategies, explain this seeming contradiction. Generalize.

For convenience, we rewrite the KE:

$$\dot{x} = -\frac{w_1}{R} y\phi + w_2 \sin \psi$$

$$\dot{y} = \frac{w_1}{R} x\phi - w_1 + w_2 \cos \psi, \qquad -1 \leqslant \phi \leqslant 1$$

where

$w_1 = P$'s speed

$w_2 = E$'s speed

$R =$ minimal turn radius of P.

For $\mathscr{C}$ we have

$$x = l \sin s, \qquad y = l \cos s$$

and $\mathscr{E}$ is the x,y-plane exterior to this circle.

The many preceding similar problems will render details of the integration unnecessary. We state some results without proof.

The useable part of $\mathscr{C}$ is

$$-s_0 < s < s_0 \tag{10.3.1}$$

where $\cos s_0 = w_2/w_1$, $\sin s_0 > 0$.

[5] Section 8.8.

The optimal paths are given by

$$x = (l - w_2\tau)\cos\left(s + \frac{w_1}{R}\tau\right) + R\sin\frac{w_1}{R}\tau$$

$$y = (l - w_2\tau)\sin\left(s + \frac{w_1}{R}\tau\right) + R\left(1 - \cos\frac{w_1}{R}\tau\right).$$

where s satisfies (10.3.1) and $0 \leqslant \tau \leqslant l/w_2$.

The curves of constant V are arcs of circles with centers $R(1 - \cos(w_1/R)\tau)$, $R\sin(w_1/R)\tau$ and radii $l - w_2\tau$. Their envelope is an arc of $\mathscr{B}$, which is also the optimal path with $s = s_0$.

Exercise 10.3.1. Prove these statements.

Research Problem 10.3.1. Although Figure 10.3.1 is drawn for the case where the barriers do not meet, our reasoning, of course, applies to the case where they do. How then does the point A, where the paths coalesce ($\tau = l/w_2$), relate to the point of intersection of the barriers? In other words, when the barriers meet and there is but a small bounded capture zone, is this filled by primary paths or does a universal curve and its tributaries cover part of it?

10.4. THE UNIVERSAL CURVE AND ITS TRIBUTARIES

We have already seen in Example 7.13.2 that the y-axis above C is a universal surface and the only[6] such. It and its tributaries correspond to the most obvious and general type of play. As redepicted in Figure 10.4.1a, P first turns sharply as possible until he is pointed at E and then travels straight; throughout E traverses the same straight path which lies on a tangent through his initial position to the nearer of P's initial curvature circles. Such was proved in Example 7.13.2.

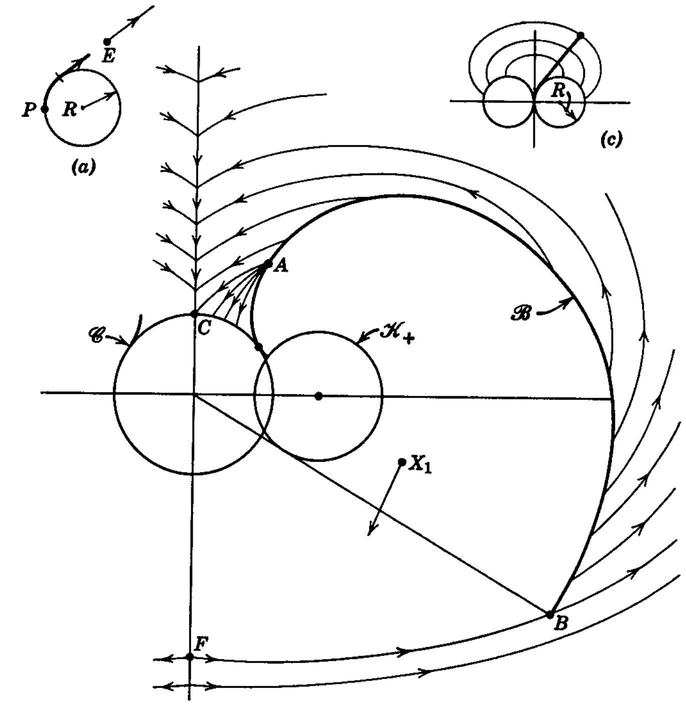
LEMMA 10.4.1. In the domain of the tributaries, the curves of constant V are involutes of the pair of turning circles.[7]

(By "pair" we mean that if the involute is drawn in the usual way by an enwrapping string, the union of the two circles can serve as the enwrapped solid guide as shown in (c) of the figure.)

Proof. We know that E's flight direction is along the tangent, here the generating "string." From the ME_1 we know that his optimal direction is normal to the curves of constant V. A simple principle of differential equations now confirms our result.

[6] An inverted sort of exception will appear in Section 10.9.

[7] The circles of centers $(\pm R, 0)$ and radius R.



(b)

Figure 10.4.1

An appendix to this chapter contains some of the analysis relating to the domain of the tributary paths.

Research Problem 10.4.1. Do the optimal paths here—the tributaries to the universal curve—admit of such a simple geometric interpretation?

The optimal paths are sketched in (b) of the figure. One of them, that which passes through B, the barrier end, meets the lower y-axis at F. From each point of the axis on or below F two optimal paths, right and left, emanate. These points thus constitute a dispersal curve. In the realistic space they correspond to positions at which E is directly behind P. The simple type of play (as at (a) in the figure) is optimal, but both players are confronted with a left-right choice and the problem of out-guessing the other. It is a typical instance requiring an instantaneous mixed strategy as discussed in Chapter 6.

On Figure 10.4.1b is indicated all covered thus far of the solution of the homicidal chauffeur game when the barriers do not meet. But we are challenged by a large empty region of $\mathscr{E}$ untouched as yet by any phase of our solution. It is roughly bounded above by $\mathscr{B}$ and the nonuseable part of $\mathscr{C}$ and below by arc of the path BF. From starting points therein—X_1 on the figure is typical—we are intuitively led to believe that a swerve must follow. If this is so, for a state such as X_1, P must begin with a sharp left turn, which will occasion a velocity of X_1 somewhat as shown by the arrow. It seems safe to conclude that ultimately $\mathbf{x}$ will cross BF and will be in a domain already treated. But what of the transition? This proves to be far from a simple question and to it we now turn.

10.5. EQUIVOCAL SURFACES

They are singular surfaces of type $(+, u, -)$. In this section we shall describe the general conditions for the occurrence of an *equivocal surface* (abbreviated ES), but our later, more refined analysis will be limited to the case where $n = 2$. In fact, a single example will suffice to elucidate the whole theory.

The names derives from the choice of two optimal strategies at each point of the surface at the disposal of one of the players.

These surfaces, unlike many types already studied, have no counterpart in the calculus of variations. They cannot occur in a one-player game. The implication is that the underlying theory of differential games must essentially depart from any mere extension of classical ideas.

Suppose, in a certain differential game, that it is clear, on intuitive or other grounds, that the optimal paths behave as typified in Figure 10.5.1. Those that reach $\mathscr{C}$ belong to some family (1), but at (2) there is an entirely different family. Optimal play demands that $\mathbf{x}$, starting as shown, must first traverse a path (2) and then switch to a path (1). The paths (1) are yielded by a valid solution of the main equation which extends well up to and beyond where a juncture with paths (2) would be expected to occur.

We are interested in the mechanism of the transition from one type path to the other.

For convenience we shall refer to the paths (1) as *primary* and their union as the primary domain; the paths (2) will similarly be *secondary*.

Let $\mathscr{S}$ be the surface at which the transition occurs (on the figure it might be one of the dashed curves). On reaching $\mathscr{S}$ via a secondary path, $\mathbf{x}$ must either penetrate $\mathscr{S}$, remain on it, or regress backward. The third alternative is absurd if the secondary paths are posited as optimal and the bearers of unique optimal strategies in their domain.

The second alternative, depicted at (a), has $\mathscr{S}$ acting as a semiuniversal surface: once $\mathbf{x}$ reaches $\mathscr{S}$, $\mathbf{x}$ traverses it. We will refer to values of the control variable which engender such play as *traversing strategies*.

In the remaining alternative ((b) in the figure), $\mathscr{S}$ acts as a transition

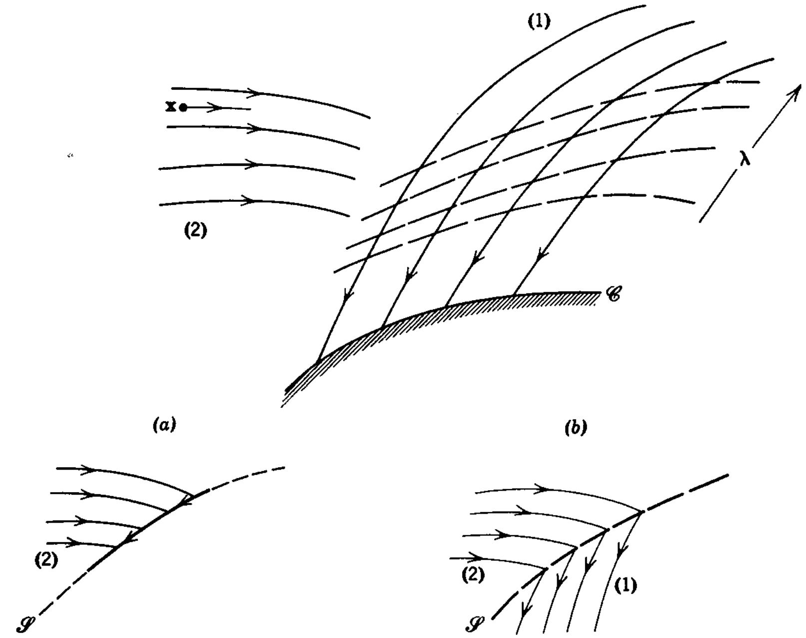


Figure 10.5.1

surface: $\mathbf{x}$ penetrates $\mathscr{S}$. We will speak here of *penetrating strategies.*

We will now make four explicit assumptions:

A1. The choice of *which* surface is to act as $\mathscr{S}$ rests entirely with one of the players. For definiteness let us say he is P.

A2. The choice as to whether $\mathbf{x}$ penetrates or remains on $\mathscr{S}$ rests with the other player. Calling him E, then he alone chooses between the penetrating and traversing strategies.

Now let us consider some "smooth" one-parameter (λ = the parameter) family of surfaces, each eligible to be $\mathscr{S}$. By "smooth" we mean that distinct members of family do not intersect, and as λ increases, the surfaces shift smoothly in the same general direction. (See the set of broken curves in the figure.)

A3. Starting from any fixed $\mathbf{x}$ in the secondary domain, let E elect a traversing strategy which is optimal in all other aspects at his disposal. Let P choose various λ (member of our family of possible $\mathscr{S}$) but otherwise act optimally. *Then the payoff is a decreasing*[7] *function of* λ.

A4. Under the same hypotheses, except that now E elects a penetrating strategy, *the payoff is an increasing*[8] *function of* λ.

[8] If (without changing the parametrization) *both* of these rates are reversed, the conclusions we soon will draw from these assumptions remain valid.

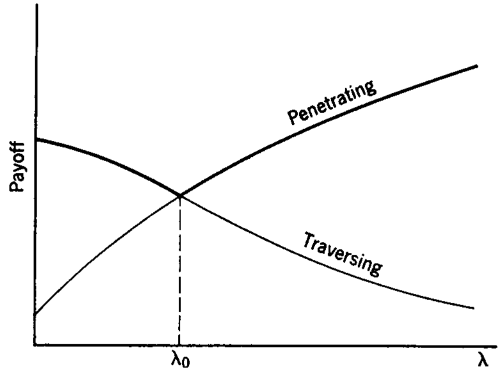


Figure 10.5.2

Plots of these two payoffs, as functions of λ, have the appearance then of Figure 10.5.2. As E is the maximizing player, for each λ his choice between the traversing and penetrating strategies will be that yielding the higher payoff. Thus it will be represented by some point on the composite upper curve (overscored in the figure). Consequently P will pick the λ (λ_0 in the figure) which yields the minimum of this upper composite curve, which will be at the intersection.

We draw the conclusion, as this situation prevails for all smooth families and all admissible starting points in the secondary domain, that the optimal surface $\mathscr{S}$ must enjoy the property:

> *From each point of $\mathscr{S}$ optimal play of the traversing or penetrating type each leads to the same payoff, the common value being the Value.*

This requirement will be known as *ES condition* and a surface fulfilling it, an *equivocal surface.*

For $n = 2$, the only case which we have studied in any detail, it turns out that the ES condition is tantamount to an ordinary, first-order differential equation. Thus, generally, an equivocal surface can be passed through an arbitrary point. What starting points befit a particular game is often clear from its context; generally we might observe a possible analogy between this question and that of fitting a barrier (Section 8.5).

Still retaining $n = 2$, let P and E have roles as in the assumptions. We shall suppose that P has a linear vectogram and his optimal strategies in the primary and secondary domains use the two extreme values of ϕ. (It is hard to see how anything else is possible.) To navigate the equivocal surface, P will require an intermediate $\phi = \breve{\phi}$. Further, we shall assume that time of capture is the payoff ($G = 1$) although it would not be hard to extend the following results to arbitrary positive G. We also suppose vectogram additivity, that is, the resultant velocity of **x** is the sum of the two players' choices.

LEMMA 10.5.1. Under the above circumstances, the optimal traversing strategy of E on an equivocal surface[9] is that which maximizes the velocity component perpendicular to P's headline in a direction counter to that of **x**'s travel on the surface.

Proof. With $G > 0$, it is clearly best for E, when he permits traverse of the equivocal surface, to keep the speed along it as low as possible. In Figure 10.5.3 let X be a position on the equivocal surface ES, XA_1 and XA_2 be P's extreme velocity vectors (A_1A_2 is his headline), and let the dashed line be tangent to ES at X. Whatever be his vectogram, suppose that E selects a velocity A_1B_1 (or its translates A_2B_2 or A_3B_3). Now ϕ must yield a resultant tangent to ES; thus P selects XA_3 such that the resultant XB_3 lies on the tangent. Then, to minimize $|XB_3|$, E strives to make the line B_1B_2 as far from A_1A_2 as possible. But such requires the strategy of the lemma.

When the equivocal surface is known, then so too will be V on it; the secondary paths are constructed in the usual way, using this data as initial conditions. The following lemma shows that the initial V_i are reckoned with E playing $\breve{\psi}$, the optimal traversing strategy of the above lemma (and of course, P playing one of the extreme values of ϕ).

LEMMA 10.5.2. The optimal traversing strategy of E on an equivocal surface is continuous with his optimal strategy on the secondary paths.

Proof. Suppose the lemma false; let E play his optimal strategy with the alleged discontinuity and let P play thus: recalling A1 (choice of which

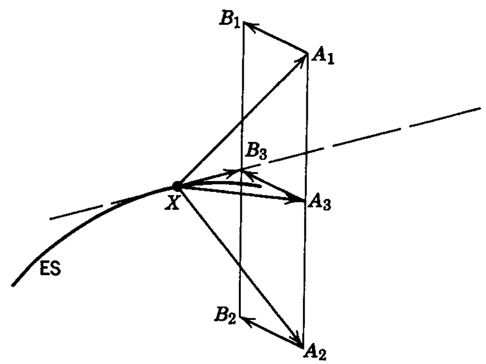


Figure 10.5.3

[9] More appropriately "curve" when $n = 2$.

curve is to act as ES resides with P) let him choose one, $\mathscr{S}'$, close to the actual ES on the "secondary" side. That is, P makes his switch from an extreme ϕ to $\breve{\phi}$ slightly *before* $\mathbf{x}$ reaches the equivocal surface. He can do so with (at worst) an arbitrarily small penalty in payoff.

But as $\mathscr{S}'$ lies in the secondary domain, due to the discontinuity, E's opposition to the travel of $\mathbf{x}$ along $\mathscr{S}'$ will be bounded away from the optimum of Lemma 10.5.1. Thus, if P's defection above is sufficiently small, he will obtain a payoff better than the Value. But such is impossible against an opposing optimal strategy.

Problem 10.5.1. Give a formal proof according to the following outline. For the KE take

$$\dot{x}_i = \alpha_i(\mathbf{x})\phi + \beta_i(\psi, \mathbf{x}), \qquad (i = 1, 2).$$

The initial conditions for the V_i on the ES (parametrized by τ) result from solving

$$\sum_{i=1,2} V_i[\alpha_i(\mathbf{x})\breve{\phi}_i + \beta(\breve{\psi}, \mathbf{x})] = -1 \qquad \text{(ES condition)}$$

and

$$\breve{\phi} \sum_i V_i\alpha_i + \max_{\psi} \sum V_i\beta_i = -1 \qquad \text{(ME)}.$$

Show that a solution can be obtained through

$$\sum_i \alpha_i V_i = 0$$

$$\sum_i \beta_i(\breve{\psi}, \mathbf{x})V_i = \max_{\psi} \sum_i \beta_i V_i = -1.$$

(To apply Lemma 10.5.1, observe that $\beta_1\alpha_2 - \beta_2\alpha_1$ is the requisite normal velocity.)

10.6. AN EXAMPLE WITH AN EQUIVOCAL SURFACE: PRELIMINARIES

Can a differential game actually have an equivocal surface as part of its solution? If so, in a case such as sketched in Figure 10.5.1, then part of the primary solution would have to be abruptly cut off even though the reject is part of a formally valid construction. This matter merits close scrutiny which we bestow on this aspect of the example below.

A second dividend accrues from study of this example. It emulates that part of the homicidal chauffeur concerning passage around the end of the barrier. It does so closely enough to warrant our applying the conclusions drawn here directly to our main subject game.

Example 10.6.1. A game with an equivocal surface. We take $\mathscr{E}$ to be the upper half-plane ($y \geqslant 0$) and $\mathscr{C}$ to be the positive x-axis:

$$\mathscr{C}\colon\ x = s \geqslant 0, \quad y = 0$$

The payoff will be time to termination. The vectogram for P will be as in Figure 10.6.1. The vertical component will be bounded by the constants $\pm b$, whereas the horizontal component will be $u(y)$, a smooth, increasing, and positive function. On the other hand, E will have a circular vectogram of fixed radius w. We shall require that $b > w > u(0)$ and that for some, necessarily unique, y_0

$$w = u(y_0).$$

From Example 8.4.3 we know the barrier $\mathscr{B}$. It is shown as the arc OB in Figure 10.6.1b, the coordinates of B being x_B, y_B with $y_B = y_0$. Note that at B, $\mathscr{B}$ is vertical; the barrier terminates just as in the homicidal chauffeur game—when its tangent is parallel to P's headline.

The least time paths, when starting positions are rightward enough to permit unhindered access to $\mathscr{C}$, are easy. Since all vectors are independent of x, P and E simply maximize and minimize their downward vertical velocity components. That is, P selects his lower extreme vector and E's

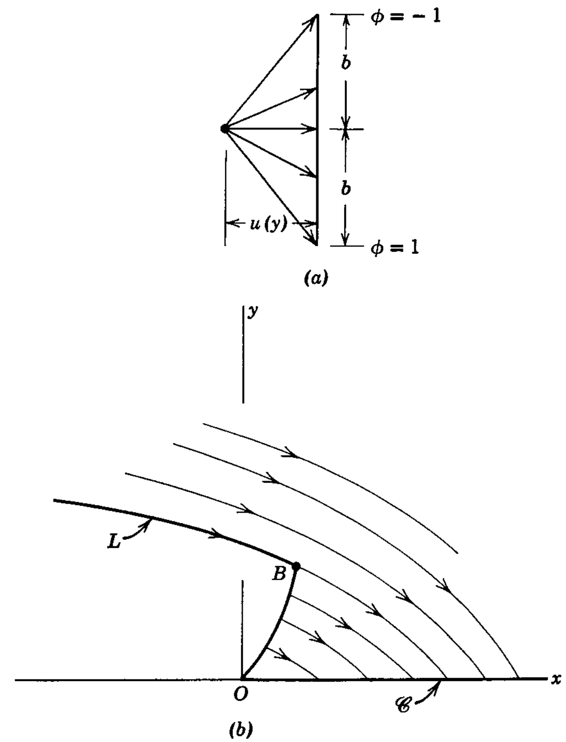


Figure 10.6.1

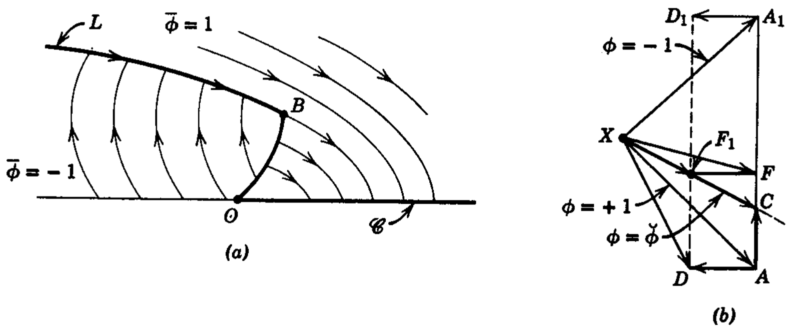


Figure 10.6.2

velocity points straight up. The resulting paths are all translates of the same curve and are sketched at (b) of the figure. Let L be the arc of that one passing through B which lies left of B. Those paths which are not terminated by $\mathscr{B}$ constitute the primary paths in the sense of the last section. In the primary domain it is clear that (consider the vertical velocity component),

$$V = \frac{y}{b - w}. \tag{10.6.1}$$

Exercise 10.6.1. Calculate the full solution in the primary domain when

$$b = 3, u = y + 1, w = 2$$ [10]

At this point one might suspect that the solution could be completed by taking initial conditions on L and obtaining the optimal paths which feed into it from below. That is L is to be a semiuniversal curve and the optimal paths to appear as in Figure 10.6.2a.

Such a construction can actually be carried out. On L, H is to be the primary V as given by (10.6.1), and P selects his upper extreme velocity vector. The analysis is not difficult.

Problem 10.6.1. Prove that a solution as in Figure 10.6.2a can be constructed and/or actually construct it for the specific data of Exercise 10.6.1.

THEOREM 10.6.1. The preceding construction does not supply the correct solution.

Proof. Suppose it did. Then P's optimal $\phi = \tilde{\phi}$ would entail the choice of his lowermost velocity vector for $\mathbf{x}$ on or above L and his uppermost when below. On L, as we have seen, the optimal $\psi = \bar{\psi}$ leads to a vector pointing straight up for E.

[10] The barrier with this same data was the subject of Exercise 8.4.3.

Diagrammed, these strategies appear as in Figure 10.6.2b, where X is a point of L. The above strategies lead to the velocities XA for P and AC for E. Thus the resultant XC is tangent to L.

We will now have E play an alternate strategy. In some neighborhood N of L, E chooses a velocity vector pointing due left, which appears as AD in the figure. Let P play a K-strategy (Section 2.6) with $\tilde{\phi}$ as tactic and finely enough subdivided so that all his points of decision fall within N.

For X on L, P will thus choose XA. The resultant is now XD, which brings $\mathbf{x}$ (or X) below L. Thus P's next decision will invoke the upper vector XA_1 yielding XD_1 as resultant velocity. Ultimately $\mathbf{x}$ will again be brought on or above L and the resulting velocity returns to XD.

Thus $\mathbf{x}$, by an alternation between velocities XD and XD_1, follows a path oscillating about L. With finer subdivisions, such, it is easy to see, approaches an equivalent motion with velocity XF_1, the convex linear mixture of XD and XD_1 which is tangent to L. That is, the payoff becomes as if P had played the intermediate ϕ resulting in velocity XF. (FF_1 is horizontal.)

But $|XF_1| < |XC|$. Thus $\mathbf{x}$ traverses L more slowly under the alternate strategy and the payoff is increased. Thus the type of play pertaining to (a) of the figure is not optimal, for E can do better than the payoff it yields when opposed by P playing all sufficiently fine K-strategies with $\tilde{\phi}$ as tactic.[11]

COROLLARY 10.6.1. Not all of the primary solution is valid.

Proof. If the "solution" above L were correct, we could formulate a new game for which $\mathscr{E}$ is bounded by $\mathscr{B}$, L, and the negative y-axis. Here $\mathscr{B}$ and L are to form $\mathscr{C}$, the usable part of which is clearly L. Further, H is to be the primary Value on L while G is again 1. The actual solution to this new game results from the construction whose existence we asserted (and requested in Problem 10.6.1). Then the solution of the original game would have to be just what we proved it was not.

Before continuing with this example, let us observe that essentially the foregoing construction is valid in similar one-player games.

Example 10.6.2. The same as Example 10.6.1, except that now $w = 0$. The KE are now

$$\dot{x} = u(y)$$
$$\dot{y} = -b\phi, \qquad -1 \leqslant \phi \leqslant 1.$$

There is now no barrier. But L persists; it is now the path of the first stage solution leading to the origin. Adapting the preceding ideas, we are led to a construction having paths as at (a) of Figure 10.6.3. In view of

[11] Such a strategy appears to be the only way we can assign a meaning to his playing $\tilde{\phi}$.

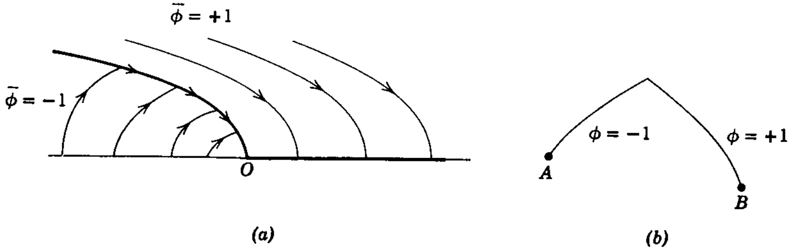


Figure 10.6.3

Theorem 10.6.1, we give a brief but direct proof that the constructed solution is correct.

Clearly P should play his lower extreme velocity ($\phi = 1$) throughout the whole partie whenever this brings **x** to $\mathscr{C}$. For we can reckon the payoff in terms of the vertical velocity component only, that is, consider the time required for the projection of **x** on the y-axis to descend to $\mathscr{C}$. But $\phi = 1$ yields the maximal possible downward speed throughout play and therefore is certainly best. Thus the primary stage paths are correct.

Suppose now **x** is to travel from a given point A to another B to the right of A but too distant to admit the preceding strategy. The path of least time is depicted at (b); the first part has $\phi = -1$ and a second $\phi = 1$. Exactly one such path exists, for if we draw the $\phi = -1$ path through A and the retrograde path with $\phi = 1$ through B, they will intersect just once. To see that this path is optimal, we think this time in terms of the horizontal velocity component. It is greater with increasing y and clearly the path at (b) attains the maximal integrated y.

Thus if the starting point lies to the left of L, the least time path *to O* is of the type asserted and shown at (a). Finally, it cannot be that P can do better if **x** first reaches $\mathscr{C}$ at some point rightward of O. For were this the case, the path would have to cross L; at the first such instant it is under the aegis of the primary strategy and follows L to O.

Problem 10.6.2. Show that if $u = y - 1$, and so not always positive, that not all of L acts as a seat of initial conditions for the second stage paths.

Problem 10.6.3. Still with $u = y - 1$, if $\mathscr{E}$ is enlarged to the entire plane, but $\mathscr{C}$, save possibly at O, must be approached from above, show that there are "third stage" paths.

10.7. AN EXAMPLE WITH AN EQUIVOCAL SURFACE; SOLUTION

We return now to Example 10.6.1. We are going to show that the assumptions A1 to A4 of Section 10.5 are satisfied. As a prelude we

observe that some parts of the first stage paths are valid; on the other hand, in some region below and left of B the optimal $\bar{\phi}$ must be -1.[12] Our problem is the transition between these phases.

A1. Clear. For P can pick the curve which demarcates the regions where he takes $\phi = 1$ and $\phi = -1$. If this curve is to function as an equivocal one, of course he must use an intermediate $\phi = \check{\phi}$ on it. But from the standpoint of K-strategies, this is unnecessary. As we will see under A2 below, the ES-strategy of E compels a zigzagging about the ES somewhat as in the proof of Theorem 10.6.1.

A2. The curve that we shall refer to as the ES in this paragraph can be any smooth one such that

1. it starts at B.
2. it cuts each primary path above B with steeper slope so that the equivocation in the definition of an ES is possible.
3. it can be navigated by **x** when E plays $\check{\psi}$, the optimal traversing strategy, and P plays *some* intermediate ϕ which we shall call $\check{\phi}$.

We will show that E dictates the choice between traversing and penetrating play.

The various possibilities for a vectogram for X_0 a point on the ES are shown in Figure 10.7.1. From 2, X_0 is above B and so $u > w$. Here A_1A_2 is P's headline. As the optimal primary (and also penetration) strategy calls for E's erect velocity vector A_1A_3 and P's primary optimum is $\phi = 1$ or the vector X_0A_1, the resultant velocity X_0A_3 is tangent to the primary path. From 2 the tangent to the ES has a steeper slope and is shown as the dashed line. From Lemma 10.5.1, the optimal traversing strategy of E yields the vector A_1B_1 (or A_2B_2 or A_4B_4) perpendicular to A_1A_2. Thus $\check{\phi}$ leads to the vector X_0A_4 by P so that the resultant X_0B_4 is tangent to the ES.

Now suppose E should switch to A_1A_3, (penetration strategy). The resultant velocity choice left to P ranges from X_0A_3 to X_0A_5. All carry **x** above the ES into the primary domain. From here, if each player wants to reap the Value, he must play optimally and X_0A_3 results. Thus the partie concludes in the primary domain.

Now suppose E adheres to the traversing strategy A_1B_1. We will show that P cannot deviate from $\check{\phi}$ without incurring a loss in payoff.

Suppose for some short interval he used a $\phi < \check{\phi}$. Then **x** would rise a positive distance above the ES, and the primary optimal strategies would prevail for both players. By thinking of vertical velocity components only, we see that this route, where **x** rises and then descends again, takes more

[12] Upper extreme velocity.

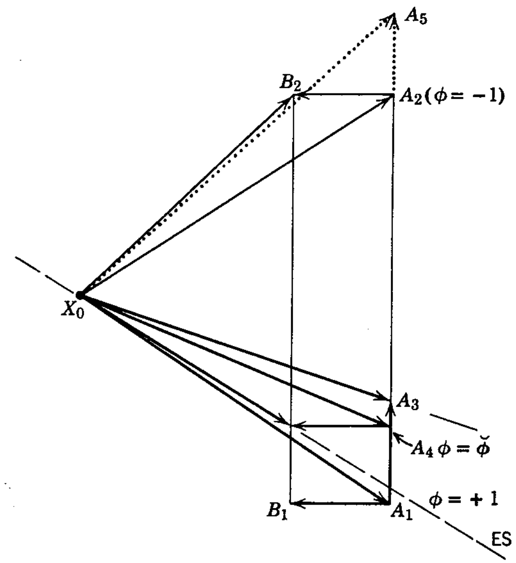


Figure 10.7.1

time than a direct primary descent from X_0, and therefore more time than $V(X_0)$.

On the other hand, if P adheres to a $\phi > \breve{\phi}$ during some interval, $\mathbf{x}$ descends below the ES. (By Lemma 10.5.2, E keeps close to the velocity A_1B_1.) At some later time $\mathbf{x}$ must recross the ES, say at X_1. For, by 1, the ES and $\mathscr{B}$ form one curve separating $\mathbf{x}$, after its descent, from $\mathscr{C}$ and E can prevent $\mathbf{x}$ from crossing $\mathscr{B}$. Let E adopt a strategy which does so prevent, yet which otherwise retains A_1B_1 when $\mathbf{x}$ is below the ES. Now the depressed route from X_0 to X_1 takes longer than the one via the ES. We can see this by considering horizontal velocity components only and recalling that $u(y)$ is increasing. Thus there is a way for E to force P to do worse than the Value.

To deal with A3 and A4 we take for our possible $\mathscr{S}$ a family of curves each fulfilling 1, 2, 3 as described under A2. Let λ be as in Figure 10.7.2a.

A3. We are to show that, when E plays the traversing strategy, the payoff decreases with increasing λ. It suffices to show that if (see (b) of the figure) AB and DB are arcs of $\mathscr{S}$ curves, the time for $\mathbf{x}$ to travel ADB is less than that for travel over the curve AB. Of course, AD is part of a "secondary" optimal path.

We may concentrate on horizontal velocity components. On AB and DB such are $u(y)-w$ and for AD it $\geqslant u(y)-w$. As u increases with y, the

higher path ADB everywhere engenders the greater horizontal velocity and so consumes less time of traverse. Thus the payoff decreases with λ.

A4. We now show that if $\mathbf{x}$ penetrates $\mathscr{S}$ to the primary domain, the payoff increases with λ. From (10.6.1), V is an increasing function of y in the domain. Thus, as the $\mathscr{S}$ curves have negative slope, the greater λ, the more time for $\mathbf{x}$ to reach $\mathscr{S}$ *and* the greater V once $\mathbf{x}$ gets there.

Thus the existence of an equivocal surface is established. We now turn to its construction. The ES condition states generally that for travel along this surface

$$\frac{dV}{dt} = -G(= -\mathring{V}) \tag{10.7.1}$$

where V is the Value in the primary domain. In our example we shall work retrogressively, $G = 1$, and V is given by (10.6.1). But first we must write the KE:

$$\dot{x} = u(y) + w \sin \psi$$
$$\dot{y} = -b\phi + w \cos \psi, \qquad -1 \leqslant \phi \leqslant 1.$$

From Lemma 10.5.1, $\sin \breve{\psi} = -1$, $\cos \breve{\psi} = 0$, and (10.7.1) becomes in our case

$$1 = \mathring{V} = V_x\mathring{x} + V_y\mathring{y} = 0 + \frac{1}{b-w}\, b\breve{\phi}. \tag{10.7.2}$$

Thus for the ES

$$\breve{\phi} = 1 - \frac{w}{b} \tag{10.7.3}$$

which, when inserted in the KE, furnish its differential equations

$$\begin{aligned} \mathring{x} &= -u(y) + w \\ \mathring{y} &= b - w. \end{aligned} \tag{10.7.4}$$

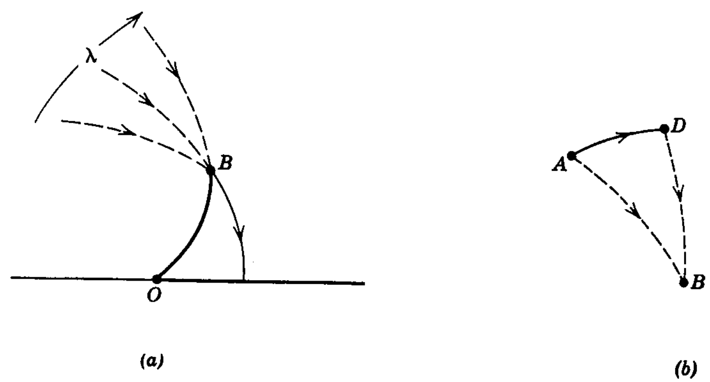


Figure 10.7.2

Our equivocal curve is the integral of this system of differential equations beginning at B: for initial conditions take $x = x_B$, $y = y_B = y_0$ when $\tau = \tau_B = V(B) = y_0/(b - w)$ from (10.6.1). Thus τ on the ES is V.

Note that at B, $\mathring{x} = 0$, $\mathring{y} > 0$ so that the equivocal curve here has a vertical tangent. We learned in Example 8.4.3 that the same was true of the barrier. Thus the two curves join smoothly at B.

Research Problem 10.7.1. Is this a general truth? When an equivocal surface attaches to the termination of a barrier, do the two surfaces meet smoothly?

Exercise 10.7.1. Using the data of Exercise 10.6.1, show that in this case the ES is a half-parabola.

[In fact, its equations are

$$x = -\tfrac{1}{2}(\tau - 1)^2 + x_B \qquad y = \tau \tag{10.7.5}$$

for $\tau \geqslant 1$.]

We conclude this section by partially treating the secondary paths—those emanating *from*[13] the equivocal surface—with an eye to corroborating the material of Lemma 10.5.2.

The ME_2 of the current problem is

$$u(y)V_x - b\bar{\phi}V_y + w\rho + 1 = 0$$

where

$$\bar{\phi} = \operatorname{sgn} V_y \text{ and also } \sin\bar{\psi} = V_x/\rho,\ \cos\bar{\psi} = V_y/\rho,\ \rho = \sqrt{V_x^2 + V_y^2}.$$

The RPE are

$$\mathring{x} = -u(y) - w\frac{V_x}{\rho}, \qquad \mathring{V}_x = 0$$

$$\mathring{y} = b\bar{\phi} - w\frac{V_y}{\rho}, \qquad \mathring{V}_y = u'(y)V_x.$$

For initial conditions we use the equations of the ES with, for notational consistency, τ replaced by s. Then on the ES, $V = s$. Using (10.7.4) for x_s and y_s, our usual

$$V_x x_s + V_y y_s = V_s$$

becomes in this case

$$-V_x(u - w) + V_y(b - w) = 1 \tag{10.7.6}$$

which is to be solved for V_x and V_y in conjunction with the ME_2. We can

[13] In the retrograde sense.

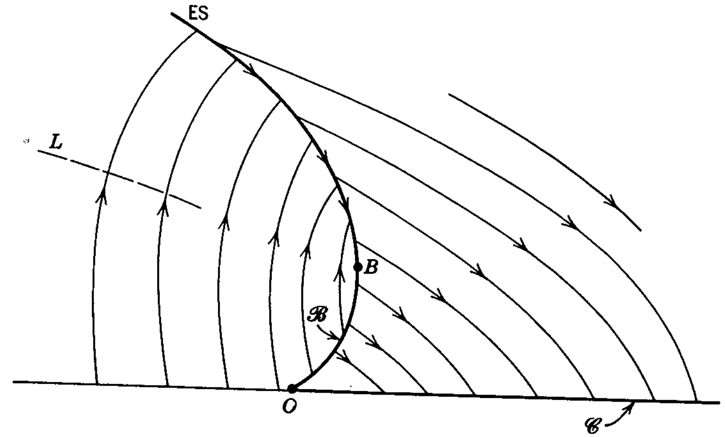


Figure 10.7.3

see at once that a suitable formal solution is

$$V_x = \frac{-1}{u - w}, \qquad V_y = 0. \tag{10.7.7}$$

We must note that $V_x < 0$ when $y > y_0$ or on all the ES but B.

To ascertain $\bar{\phi}$, we must look at $\mathring{V}_y$ on the ES. From the RPE

$$\mathring{V}_y = u'V_x < 0$$

as $u' > 0$, $V_x < 0$. Thus $\bar{\phi} = -1$ or P uses his uppermost vector as we should expect him to.

From (10.7.7) we see that initially ($\tau = 0$) $\sin\psi = -1$. Thus E's velocity is left and horizontal; it is continuous with the ES strategy as Lemma 10.5.2 asserts.

The remainder of the problem entails little innovation and we leave it to the reader.

Exercise 10.7.2. For the data of Exercise 10.6.1, show that the optimal tributary paths to the ES are given by

$$x = (1 - s)\tau + \tfrac{3}{2}\tau^2 - \tau\sqrt{1 + \tau^2} + \log(\tau + \sqrt{1 + \tau^2}) - \tfrac{1}{2}(s - 1)^2 + x_B$$
$$y = -3\tau + 2\sqrt{1 + \tau^2} - 2 + s.$$

Finally, the picture of the optimal paths is something like Figure 10.7.3.

10.8. DISCUSSION OF EQUIVOCAL SURFACES

Can the equivocal surface, as a smooth adjunction to the barrier, be regarded as an extension of the latter? Only in the exiguous sense that each player can make $\mathbf{x}$ penetrate the surface only at the cost of a penalty in payoff.

It is interesting to see how the players would be bound by such proscriptions in practice, that is, in some realistic game for which an adequate model is like the present example.

Suppose at a certain point of an (optimal) partie, $\mathbf{x}$ reaches the equivocal surface and E elects the traversing strategy. Continued optimal behavior by P retains the ES as a path and ultimately $\mathbf{x}$ reaches B. Here E no longer has the dichotomy of strategies; to maximize the payoff he is bound to the primary choice. But also at his disposal is the optimal strategy of the game of kind. If he plays so, then P must retaliate with his such strategy or else $\mathbf{x}$ will penetrate the barrier and P will not be able to compel an imminent termination. To obtain a later one entails recrossing of the ES; if E repeats the tactic and P the response, continued recurrence precludes termination ever. Thus P is forced, from B on, to play the game of kind strategy; $\mathbf{x}$ follows $\mathscr{B}$ and he must be content with a neutral[14] outcome.

Is it wise for E to act so—to play the game of kind optimal strategy when $\mathbf{x}$ reaches B? It depends on how we assess the neutral outcome. We could adopt the point of view of Chapter 8 and consider neutrality as something inferior to proper interior termination. To have an exact theory we must, of course, modify the original game by specifying a numerical value of the payoff for the neutral outcome. How we so so determines the answer to the above question.

In a "practical" execution the neutral outcome lies at the brink of failure to terminate; such can be occasioned by any slight accident and P will covet a margin of safety. The simplest and most reasonable way to attain it seems to be: when $\mathbf{x}$ first reaches the ES, P continues his strategy ($\phi = -1$) for some short interval, thus bringing $\mathbf{x}$ across the ES, slightly into the legitimate primary domain.

An arbitrarily small penalty of payoff now buys P release from the dichotomy of the ES, for from now on there is single straightforward optimal strategy for each player.[15]

Thus, interesting as the ES phenomenon is in theory, the accompanying action probably is not of much importance in applications. For in realistic play it will probably be circumvented as above. But the equivocal surface remains as a boundary, delineating two different kinds of optimal play.

[14] As in Chapter 8. Here the meaning is that x touches $\mathscr{C}$ at its endpoint, the origin.

[15] Let us note that "realistically" P was liable to such a penalty even under optimal play. For if E switches from the traversing to the penetrating strategy at some unexpected instant, in "practice" there is bound to be some positive time lag in P's response.

We conclude with

Research Problem 12.8.2. If $u(y)$ has a maximum, say at y_1, then it is clear that the line $y = y_1$ will be a ϕ-universal surface sufficiently far to the left. What is the rest of the solution? In particular, do the US and ES (if there is one) meet?

10.9. THE EQUIVOCAL PHENOMENON IN THE HOMICIDAL CHAUFFEUR GAME

The ideas of the preceding sections expeditiously fit the homicidal chauffeur problem; we can construct an equivocal curve, which begins at the end of the barrier, by the same method.

Let us suppose the left and right barriers do not meet and that $\mathbf{x}$ starts from a point near and under the right barrier. Capture requires the swerve maneuver: P must force $\mathbf{x}$ downward and around the barrier. He begins by sharply turning left ($\phi = -1$) and so occasions the descent of $\mathbf{x}$. But at some later time he will be playing the strategy of the right tributaries of the universal surface, which call for $\phi = 1$, that is, he turns sharp right until E is directly in front of him ($\mathbf{x}$ on the US) whence the play ends with a direct chase along a straight line. The ES is here the locus on which P makes the transition of $\breve{\phi}$ from -1 to $+1$.

On the ES, E has the choice between the traversing and penetrating strategies. What is $\breve{\psi}$, the value pertaining to the former? From Lemma 10.5.1, we see that $\breve{\psi}$ demands that E point his velocity vector normal to P's headline. From Lemma 10.2.1, this headline is normal to the radius vector OX. Thus E's velocity directly toward O, implying

In the realistic space if E plays the traversing strategy he follows a course of pure pursuit[16] *of P.*

As for P, his optimal strategy $\breve{\phi}$ for $\mathbf{x}$ on the ES is generally distinct from 0 and ± 1. This appears to be the only optimal instance of his following a path other than straight or circular with radius R.

The procedure for ascertaining the ES parallels that of Section 10.7 and we need not repeat the reasoning. As above, $\breve{\psi}$ is known as function of x, y; we wish similar knowledge of $\breve{\phi}$. It results from the ES condition

$$\dot{x}V_x + V_y\dot{y} = -1 \tag{10.9.1}$$

where V_x and V_y pertain to the (right) tributary paths and $\dot{x}$, $\dot{y}$ mean their expressions from the KE with ψ replaced by the above $\breve{\psi}(x, y)$. If then (10.9.1) is solved for ϕ, the solution will be the sought $\breve{\phi}(x, y)$. Insertion of these values of the control variables into the KE yields a pair of ordinary

[16] That is, E always travels directly toward P.

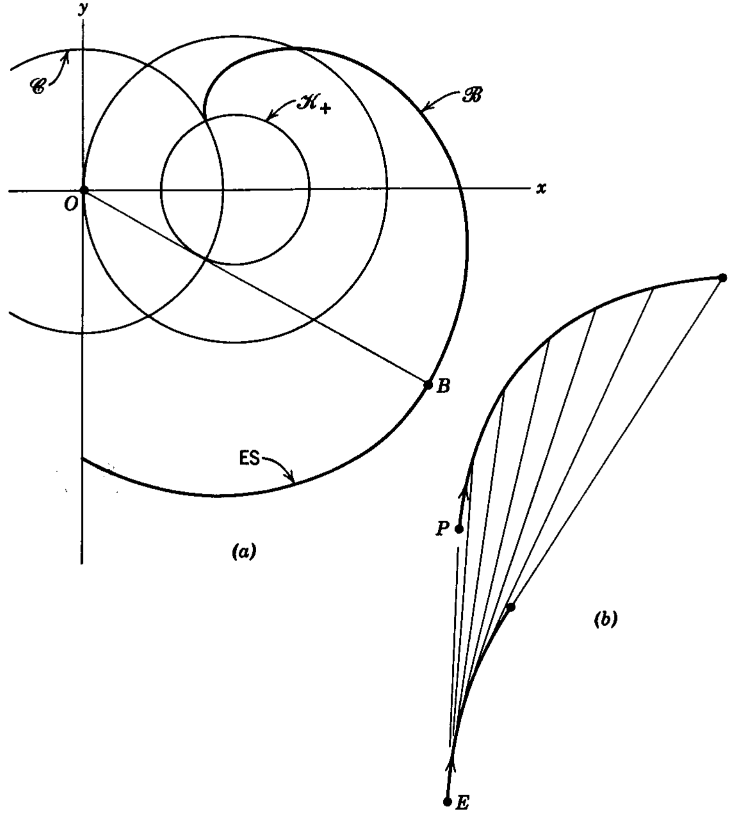


Figure 10.9.1

differential equations; their integral path which passes through B, the endpoint of the barrier, is the ES.

The foregoing analysis is done in detail in the appendix to this chapter; the final differential equations appear as (10A.9). They seem to be particularly intransigent and we have unearthed little of their specific geometric properties.

Figure 10.9.1a is an accurate construction of an instance of the equivocal curve; at (b) are similarly drawn the corresponding paths of P and E in the realistic space during this phase of the partie.

There are two possibilities for the form of the ES, depending on the parameters. It may meet the y-axis—the arc BC in Figure 10.9.2a—or it may terminate on $\mathscr{C}$ as at (b). We will discuss only the former case, which seems the more interesting.

Let us trace an optimally played partie with **x** starting at X as in (a). The optimal path meets, as shown, the y-axis, which here again acts as a universal surface. After **x** reaches it at A, **x** traverses it to C. Of course, other starting positions, such as X_1, lead to paths such as X_1D which encounter the ES directly.

It is not hard to interpret the path XAC in the realistic space. The players start from P and E of Figure 10.9.3; P begins with a sharp left turn (radius R) while E travels straight along the tangent EF to the left steering circle. After reaching F, P too travels the line EF. Such positions correspond to X on the universal AC. Note the perversion of the simpler capture depicted in Figure 10.4.1a; now E is doing the pursuing.

Both continue along the line EF, the distance between them increasing, until it equals OC, a fixed constant of the game, of Figure 10.9.2a. The positions are now P_1 and E_1 on Figure 10.9.3.

At this point P switches to his "equivocal" strategy $\check{\phi}$, and he flies the curved course P_1P_2 (see Figure 10.9.1b) while E, if he plays the traversing strategy, follows along E_1E_2 in pure pursuit. Now **x** is on the ES; at any time E may switch to his (here right) tributary strategy. That is, he will head along the proper tangent to P's right steering circle, flying away from the point of tangency, etc.

Note that when he was at P_1, P had the option of turning either right or left (**x** could follow the ES on either the right or left side of the y-axis). If E responds with the traversing strategy he points at P in either case; there is no discontinuity in ψ and P is not obliged to mix his two choices. But if E should elect the penetrating strategy, he too has a right-left choice

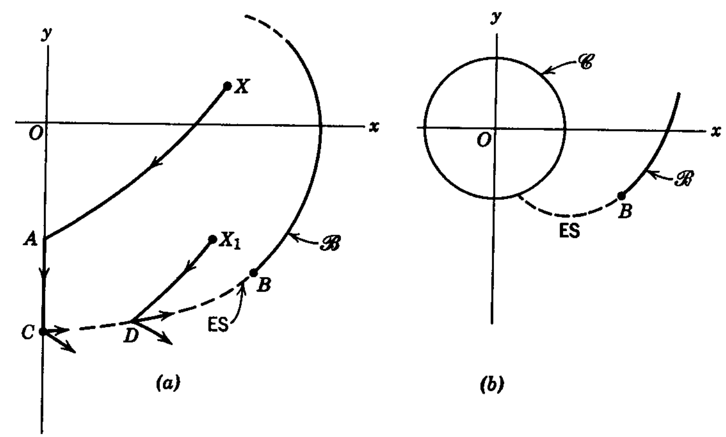


Figure 10.9.2

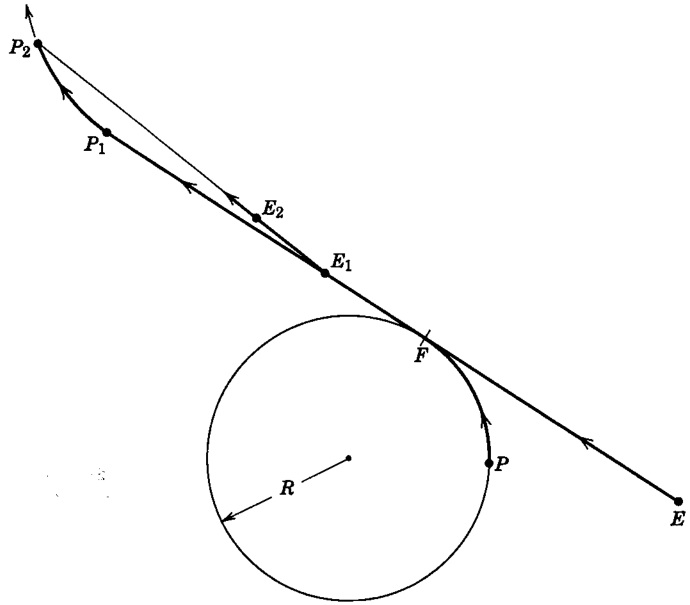


Figure 10.9.3

and an instantaneous mixed strategy is called for. Because it is safe for P to mix in either case, it should be incorporated into his optimal strategy. Recall that the part of the y-axis below C is a dispersal surface and, as mixing is required at each of these points, demanding it at C too is not an undue strain.

If E perseveres in the traversing strategy, **x** follows the ES to B, the endpoint of the barrier. Here, for maximal capture time, E must pick the penetrating strategy. But he also can play the optimal strategy of the game of kind and achieve a neutral outcome. In fact, all the discussion of Section 10.8 applies here. As there, in "practice" it would appear beneficial for P to allow **x** to pass slightly below the ES and avoid the quandary at a slight cost of payoff. In our description a convenient time to do so occurs at the positions P_1 and E_1; P merely continues along the straight course EF until the distance P_1E_1 slightly exceeds OC of Figure 10.9.2(a).

Finally, we ask if our solution is really complete. Is it possible that there can be still another phase to an optimal partie?

In Figure 10.9.4 are sketched the optimal paths prior to the ES phase in a game such as we have just discussed. Can there be a region, such as is shaded in the figure, still unaccounted for?

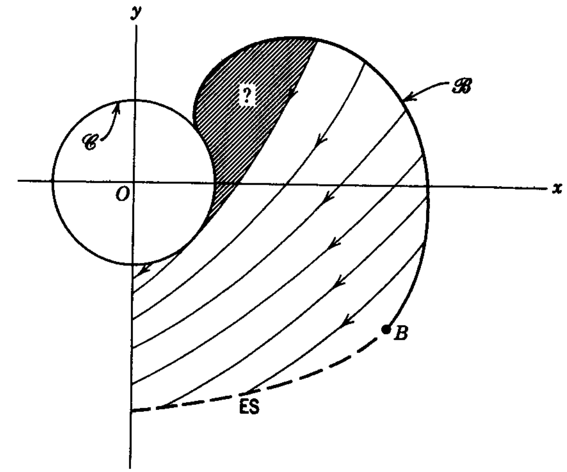


Figure 10.9.4

Research Problem 10.9.1. Answer this question. If "yes," what is the solution in the shaded area? Does there exist a further ES there, so the P begins with a sharp right turn to get **x** on it?

APPENDIX

Analytic details

We shall derive here the analytic expressions germane to the domain of the right tributary paths and the differential equations of the equivocal curve. We start by collecting the usual material.

The KE:

$$\dot{x} = -cy\phi + w_2 \sin\psi$$
$$\dot{y} = cx\phi - w_1 + w_2\cos\psi, \qquad -1 \leqslant \phi \leqslant 1$$

where $w_1 > w_2$ and $c = w_1/R$.

The ME$_2$:

$$-Ac\bar{\phi} - w_1 V_y + \rho = 0$$

where $A = yV_x - xV_y$, $\bar{\phi} = \operatorname{sgn} A$,

$$\rho = \sqrt{V_x^2 + V_y^2}, \quad \sin\bar{\psi} = \frac{V_x}{\rho}, \quad \cos\bar{\psi} = \frac{V_y}{\rho}.$$

The RPE:

$$\mathring{x} = cy\bar{\phi} - w_2\frac{V_x}{\rho}, \qquad \mathring{V}_x = c\bar{\phi}V_y$$
$$\mathring{y} = -cx\bar{\phi} + w_1 - w_2\frac{V_y}{\rho}, \qquad \mathring{V}_y = -c\bar{\phi}V_x.$$

We can observe in passing the readily computable result

$$\mathring{A} = w_1 V_x. \tag{10A.1}$$

The universal curve is given by

$$x = 0, \quad y = s, \qquad s \geqslant l$$

and as we know V on it,

$$V = \frac{(s - l)}{(w_1 - w_2)}$$

in our usual way, we complete the initial conditions with

$$V_y = V_s = \frac{1}{(w_1 - w_2)}$$

and $\qquad V_x = 0$

the latter following from symmetry and the known continuity of the V_i on a linear universal surface. On the right we clearly take $\bar{\phi} = 1$.

Integrating the RPE with these conditions yields

$$V_x = \frac{1}{w_1 - w_2} \sin c\tau, \quad V_y = \frac{1}{w_1 - w_2} \cos c\tau \tag{10A.2}$$

$$x = (s - w_2\tau) \sin c\tau + R(1 - \cos c\tau) \tag{10A.3}$$

$$y = (s - w_2\tau) \cos c\tau + R \sin c\tau$$

the latter pair being the equations of the right tributaries.

Multiplying (10A.3) by $\cos c\tau$ and $-\sin c\tau$ and adding leads to

$$(x - R) \cos c\tau - y \sin c\tau + R = 0$$

which is essentially an algebraic equation for, say, $\cos c\tau$. We solve it using the continuous root such that $\tau = 0$ when $x = 0$, giving us

$$\cos c\tau = \frac{-R(x - R) + yh}{d^2}$$

$$\sin c\tau = \frac{yR + (x - R)h}{d^2} \tag{10A.4}$$

where $d = \sqrt{(x - R)^2 + y^2}$ = distance **x** to $(0, R)$

and $\quad h = \sqrt{d^2 - R^2} = \sqrt{x^2 + y^2 - 2xR}$ = length of tangent from **x** to right curvature circle. Then, as we know τ in terms of x and y, from

(10A.3) we can easily compute s and then find V from

$$V = \frac{s - l}{w_1 - w_2} + \tau. \tag{10A.5}$$

To study the ES, direct expressions for V_x and V_y are more pertinent; they follow at once from (10A.2) and (10A.4).

We know that on the ES, E points his velocity toward O. Working retrogressively, we have E travelling directly away so that, from the KE,

$$\mathring{x} = cy\breve{\phi} + w_2 \frac{x}{r}$$

$$\mathring{y} = -cx\breve{\phi} + w_1 + w_2 \frac{y}{r}. \tag{10A.6}$$

where $r = \sqrt{x^2 + y^2}$.

The retrograde form of the ES condition demands that

$$\mathring{x}V_x + \mathring{y}V_y = 1 \tag{10A.7}$$

where the V_x and V_y pertain to the tributary paths. We substitute in (10A.7) our now known expressions for them and also make replacements from (10A.6). The resulting equation may be solved for $\breve{\phi}$, and we find after a short calculation

$$\breve{\phi} = 1 - \frac{w_2}{w_1} \frac{r^2 - xR + rh}{r(r - y)} \tag{10A.8}$$

where $r = \sqrt{x^2 + y^2}$, $h = \sqrt{r^2 - 2xR}$.

Thus we know P's optimal strategy for the ES at each point of $\mathscr{E}$ for which the constraints on $\breve{\phi}$ are not violated.

The differential equations for the ES are obtained by substituting from (10A.8) into (10A.6). They turn out to be

$$\mathring{x} = \frac{w_1}{R} y + \frac{w_2}{R} \frac{xR - yr - yh}{r - y}$$

$$\mathring{y} = -\frac{w_1}{R}(x - R) + \frac{w_2}{R} \frac{yR + (x - R)r + xh}{r - y}. \tag{10A.9}$$

The initial condition is that the integral curve is to pass through the point B. This point we know to lie on the lower ray from the origin tangent of $\mathscr{K}_+$. Now if we hold fixed all other parameters in the problem, but alter the size of capture region, the point B can be made to be anywhere, within certain limits, on the ray. Therefore for general initial conditions of (10A.9), we can take, when $\tau = 0$, a general point of the ray, which is

$$x = s\sqrt{1 - (w_2/w_1)^2}, \qquad y = -sw_2/w_1,$$

where $s > 0$, in fact, s is to be reasonably large.

Research Problem 10.A.1. Solve the differential equations (10A.9) in closed form, if this is possible. Do these equations yield any interesting geometric properties of the ES?

Research Problem 10A.2. What is the path of P, or the differential equations thereof, in the realistic space when **x** traverses the ES?

CHAPTER 11

The Application to Warfare

In certain cases, such as pursuit games, it is evident how to fit theory to practice. But in problems of broader scope, such as those dealing with battles or combat, the transition is far from obvious. The present chapter aims at clarifying such matters, explaining in general the relations between game theory and warfare, and assessing both the utility and the drawbacks. Two examples are discussed, analyzed, and partially solved; they appear both as worthy problems in themselves and as illustrations of the general ideas.

The first is the War of Attrition and Attack, concerned with the best allocation of weapons between these ends during a protracted war.

The second we have called the Battle of Bunker Hill. The central problem is the optimal allocation of firepower when two antagonists are nearing one another, the effectiveness of their weapons accordingly rising. (The title is, of course, suggested by Warren's historic command.)

11.1. GAME THEORY AND WAR

The treatment of problems of military combat by the theory of (differential and otherwise) games is a big subject and a difficult one. It is in fact large enough to warrant its own book, and this single chapter will necessarily be but the lightest of sketches. It is difficult because of the inference caried by the term *game*.

In the present chapter, the application of game theory to a problem of warfare will signify that both sides are to make effective decisions;[1] the

[1] Of course, in other contexts the set of decisions of one side may be null. Thus one-player games are really programming problems, and there is nothing wrong in treating the latter by the techniques of the former. In fact, a large proportion of the problems of applied mathematics *can* be viewed as one-player games.

objective is to determine the Value and the optimal strategies of each (in the sense of game theory). A symmetry of approach is basic to this outlook; a question of the best method of defense is just as much one of the best method of attack. This interrelationship between alternatives facing the two players—the decisions of each reckoned on the enemy's counter decisions—is the source of the increased difficulty over the simple (one-player) optimizing problem. And the increase may be vast!

Game theory is a mathematical discipline concerned with problems of conflict. Warfare, whose essence is conflict, must ultimately come under its aegis. That such has not as yet occurred to a substantial extent is due chiefly to the above increase of difficulty and the consequent dearth of methods for getting answers. To this point we return in the next section.

From now on we shall restrict our attention to zero-sum two-player games. Of course, war itself, as history shows, is almost certain to be two-player—alliances always have made it so, ultimately, if not initially—but, on the other hand, it surely is not zero-sum. However, we feel that war in the large is too vast a subject for the analyst's pencil save possibly in the remote future. But many of its less broad constituents, although not exactly so, can be depicted as zero-sum games with adequate versimilitude. For example, if we isolate an attack-defense situation, the number of weapons (bombers, missiles, tanks, troops, torpedoes) which successfully penetrate the defenses is a quantity which the attacker seeks to maximize and the defender to minimize. To ascertain the best tactics of each player and how much penetration (the Value) accrues therefrom seems a problem that may be unobjectionably modeled as a zero-sum game. In fact, the adaptability of a situation to a zero-sum approximation may be a good shibboleth for mathematical tractability.

Not every military analysis requires formulation as a game. There are some fine ones extant that furnish useful and valuable conclusions without the dichotomy of standpoint which is the heart of game theory. But in many others the embracing of a two-sided conflict is indispensable. Yet the analyst is frustrated by lack of methods. What is available to him, now and potentially?

11.2. THE AVAILABLE TECHNIQUES

The practical possibilities seem to be three.

Discrete matrix games

Such are the stuff of what now may be called the classical theory of games. The concept of a game matrix, expounded in any text on the theory, is fundamental; it was the means that enabled von Neumann to

prove the existence of optimal strategies, pure or mixed, without which the entire subject could not exist.

In principle, every finite, discrete game, as well as many other types, can be cast in the matrix form. In practice, the trouble is that the dimensions of the matrix will be astronomical unless the game is extremely simple.[2] The simplicity usually means that effectively the game is but one move long.[3] While, in theory, a sequence of moves can be embodied into a single strategy, the vast number of such is just what makes the matrix colossally unwieldy.

But there are utilitarian military instances, for example, the distribution of limited defense means over a finite set of targets of various values, with the enemy countering with an allocation of its attack strength over the same set. A second example concerns decoy attacks which may precede an authentic one.[4] The attacker is confronted with the mixed strategy of deciding on the number of feints; the defender, on allocating his munitions among the apparent raids.

Differential games

When long or continuous sequences of decisions confront the participants, a practicable mathematical solution is virtually impossible unless the game possesses an innate logical coherence in the sense we endeavored to explain in Chapter 3. Differential games is the theory of games with such a coherence. Being our subject, further discussion will fill the ensuing sections.

The greatest limitation of differential games at present to resolving military problems is its restriction to games of complete information. Possible remedial techniques and promising, if incipient, ideas are the subject of the next chapter.

Simulation

Endeavors to play table models of war games, with teams of human opponents, often assisted by computers, have been numerous and prolific in the recent (as well as the remote) past. This is not the place for a lengthy dissertation on so extensive a subject. From our point of view experiment is a means both of confirming theory and suggesting new avenues for analysis in warfare, as in other sciences. But we must not eye the parallel and lose the distinction.

As we shall stress below in regard to theory, war analysis is much looser in laws, predictability, and logic than the physical sciences. For this

[2] Tic-tac-toe (tit-tat-toe, oughts and crosses, three-in-a-row) for example, literally requires a matrix of more than 10^{300} rows!

[3] There are numerous examples in Reference [8].

[4] Reference [9].

reason simulations with elaborate and far reachingly realistic details cannot yield reliable general truths unless the partie is repeated a great many times. From the standpoint of differential games, one thing we might hope for is confirmation of conclusions of the theory. This is especially true when these conclusions spring from simplified models (such must necessarily be the case nearly always). How will they fare when imbedded in a miasma of realistic detail?

A second avenue we would advocate lies at the other extreme of playing very simple operational games designed to illuminate some specific point that has proved baffling to theory. Such, at present, will appear most likely in games of imperfect information, such as Example 12.2.4 in the next chapter.

11.3. TYPES OF APPLICATIONS

In certain cases the role of differential games in problems of warfare is explicit and does not require much comment. Such is true, for example, of most of the models involving pursuit, evasion, and similar maneuvering. We have already mentioned that a strategy is the logical equivalent to the scheme of a guidance mechanism, in the sense that is supplies instructions as to how to set the controls for each set of data measured.

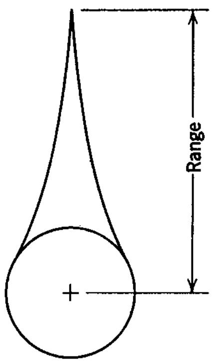


Figure 11.3.1

The utility of the results can transcend a mere application of the formal solution. For example, if the optimal guidance scheme (strategy) is too complex for practical mechanization, we would be interested in knowing how good a substitute is a simpler one. We could pit it against the optimal strategy[5] of the opponent and ascertain the penalty.

Additional instances have been noted earlier, such as applying the deadline game (Section 9.6) to channel patrol or patrol line spacing. Another is the question of whether or how much of launching boost is needed by an interceptor missile with a faster target. For a case, say, as that of the isotropic rocket (Section 9.3), if the barriers meet, the quarry cannot always be caught. But a cross section of the barrier of constant $v(= P$'s speed) supplies the set of target positions where capture is certain, no matter how the latter maneuvers, when v is the missile launch speed. The section will appear as in Figure 11.3.1. If the missile is aimed—as it should be—directly at the target, only

[5] Or, sometimes better, against a strategy deliberately designed to defeat the simple one.

the range, as shown in figure, matters. But such is the height of the crest and calculated from the simple formula derived from (9.3.8):

$$\text{Range} = \frac{v(w - \sqrt{w^2 - 2Fl})}{F}.$$

11.4. THE BROADER PROBLEMS OF COMBAT

We feel that one of the most fertile and profitable contributions of differential games to warfare lies in a broad general category of battles and combat. The path from mathematical results to usable knowledge is not a simple, direct one, and we shall endeavor below to offer guidance for its traversal.

We shall illustrate the type of ideas in mind by two examples; both shall continue in this role as we go into further detail later on. A version of the first has already appeared—Example 5.4, the war of attrition and attack—but we shall soon try to examine its central question in realistic terms. In a long-term war how best should a side apportion its attacks between the long-range objective of destroying the enemy's source of supply and the short-range one of engaging his weapons in conflict directly?

The second we have called the Battle of Bunker Hill because our central question brings to mind General Warren's famous, "Don't fire 'til you see the whites of their eyes!" When forces of any type are approaching one another, the effectiveness of their weapons increasing with the lessening distance, how best should each distribute (or conserve) his firepower? Too early action is wasteful because of the small hit probability; too late action risks the enemy's firing first and decimating too much potential fire before it is put into action.

How can such broad questions as the foregoing two be answered? Note that they are permeated by the basic concept of game theory: each side has choices to make and the merits of his decisions depend on the similar decisions made by the opponent.

The posing of the problem differs from those of the physical sciences in that the war setting is vague, complex, and unpredictable. No matter how we translate the situation into analytic terms we can never feel a comforting confidence that this is the "right" way to do it, that the answer we get will reliably concur with the actual outcome.

The best possible procedure, it appears, is to coin and then analyze an assortment of models of the game. We should start with the simplest—mathematical formulations of the situation in which the essential question to be studied stands out bold and clear, and extraneous details are eliminated. But even here there will be many possibilities; there is latitude in choosing a payoff; there is more in constructing the kinematic equations. We then strive to incorporate more realism by adding new facets. One should try to adhere only to the most essential modifications, for with addenda to the problem, the analytic difficulties proliferate rapidly.

With an assortment of solutions at hand, we should explore them for common features. It is quite possible that plausible modifications in the assumptions underlying a model can induce wide disparities in the optimal strategies. The Value, however, will generally be much more stable. Some study is now required to find what is consistent in the diversity of solutions. At times there will be remarkable features, such as a dictate that an optimal decision depends on the sign of an outré function of the state variables. When possible, we should probe such matters and try to understand them intuitively so that they can be assessed with some practical judgment. We should investigate the sensitivity of the payoff to the strategy, to our assumptions (the model), and to the values of certain coefficients appearing in the kinematic equations or in the formal payoff adopted.

Our goal should be those aspects that are independent of the details of the model. If we find such, they will be valuable pieces of information. If not, we can probably draw conclusions such as, say, that it is not critical which of a certain set of strategies is used. Or possibly, say, that a certain type of strategy is optimal only in certain circumstances, but there it pays big dividends. Generally our conclusion should be that whatever feature is common to these models should in practice be an indicator for this type of strategy.

This technique of a congeries of models accounts for our title phrase "broader problems." The results will have to apply broadly if at all. It is seldom that we will have sharp enough data for a detailed particular situation with time enough to work out the solution. But, of course, until programs of the preceding type are actually carried out, we cannot tell what will come to light.

Once results are attained, simulational gaming would be a means of testing them. We could pit a player (or team) using the results against one who is intelligent but ignorant of them and compare with plays of two knowing or two ignorant players.

The drawback to this program is that at present there seems no possibility of its execution. The armed services sponsor a vast amount of abstract mathematical research, but they appear extremely hostile to any mathematical investigations of military matters, aside, of course, from the indispensable computations needed for equipment. The private firm cannot afford to, for the number of man-hours required is considerable. The pure mathematician is hardly likely to interest himself voluntarily in

such a program; if he has a bent toward applications he will prefer remunerative work. The foundations and other pecuniary sponsors of science, with their heavy financial obligations, must necessarily be chary of backing any scientific research (of any sort) that falls outside an established category or smacks of innovation.

11.5. PROBLEMS OF FORMULATION

We now examine in a general but more detailed way some of the constituents of a program such as proposed in the previous section.

The payoff

First, we wish to make clear an obvious but often misunderstood point.[6] Whether in a game or a "programming" problem (one-player game), the term "optimal" has no meaning unless there is *one* quantity to be maximized or minimized. Often there are several important factors of merit. But it is possible to optimize only one, and before a problem can be solved it is necessary to specify which one.[7] We can take into account several at once by employing as payoff a suitably weighted linear (or other functional) combination of them. But somehow the weights must be decided and this is almost always a matter of judgment.

Now it sometimes happens that the solution is largely independent of the payoff. The strategy that captures most territory, for example, is likely to be close to the one that captures most materiel. There is no quandary then, but should the analytic results be sensitive to the payoff, and a weighted, mixed one have been used, then the results are no better than the judgment used in the weighting. A highly refined analysis, in such cases, is hardly worth while.

Of course, there are many cases where there is no quandary at all. A defense system will seek to minimize the number of penetrating enemy weapons, for example, or warning system to maximize the probability of detection. But in complex battle games, what is the logical choice of a payoff? Frequently it will not depend on the immediate foray alone but will have to be integrated with at least a portion of the enveloping war. The subsequent utility of the outcome of the battle must be the coin of the payoff.

To think of the logical ultimate, a bit of fantasy will help. Let us imagine a catalogue of military utility, which will list valuations of all sorts of military entities—weapons, munitions, personnel, bases. These

[6] We have seen armed forces study contracts which outline a complex series of requirements and then ask for optimal action.

[7] Such is, of course, the universal dilemma of human judgment. When buying a car, we can seldom optimize price, mileage, speed, and appearance at once.

utilities will change with circumstances—supply at hand and potentially, special uses for special purposes, the enemy's counter supplies. We must also have a set of rules for computing these changes. Then the payoff can be expressed in terms of total current utility of all assets that could be gained or lost by either side in the battle.

The kinematic equations

Setting them up also confronts us with alternatives. Once we have decided on the state and control variables, we must express the rates of change of the former in terms of both. It is at this point that we feel the imprecision of our subject most sharply. How are we to write definite relations between cause and effect of so indefinite a thing as a future military encounter?

There will be times when we can rely on data from experience or past analyses, but there will be times when we cannot. We will know that the greater the number w of weapons assigned against a number t of targets, the greater the damage to the latter will be. For lack of better knowledge we often assume a linear relationship,

$$\dot{t} = -cw.$$

Can we estimate the coefficient c, which means the (average) rate of destruction of targets by one weapon? Very likely, yes, although possibly coarsely.

On such a basis we can construct kinematic equations, simply or elaborately. Most or all of them will be linear in the control variables. Then the essence of the technical problem is revealed by our theory. Of key importance are the singular surfaces—universal, transition, dispersal—associated with linear vectograms. Once they are discovered, the structure of the optimal strategies is at hand.

To illustrate these ideas we now turn to the two typical problems already mentioned. As we have said already, a large amount of man-hours is needed to make practical headway, and we must be content here with the merest of beginnings.

11.6. THE WAR OF ATTRITION AND ATTACK: A STUDY

In the setting of a protracted war, each side must weigh the alternatives between direct combat and the raids on enemy sources of supply with possibly greater but future effects. Broad as such a problem is, we could think of it in still broader terms by endeavoring to admit the general problem of long- versus short-range tactics.

Example 5.4 seems to be the simplest version possible and a good one from which to start. The first kinematic equation is

$$\dot{x}_1 = m_1 - c_1\psi x_2. \tag{11.6.1}$$

Recall that x_1 and x_2 are the forces of the two sides, say amounts of aircraft, at time t; m_1 and m_2 are the unperturbed manufacturing rates; ψ and ϕ are the fractions of weapons devoted to attrition, that is, depleting the enemy's supply. The foregoing equation states that the number of P's weapons is subject to a decrease whose rate is proportional to ψx_2, the number of weapons E devotes to this purpose at time t.

Here is an instance of the linear assumption as mentioned in the last section. The meaning of c_1 in terms of, say, aircraft is the reduction of P's rate of aircraft production due to an attack by one enemy plane. Is there any hope of rationally estimating c_1 (and, of course, c_2)?

Actually c_1 is the result of many quantities—the capability of the attacking craft, the nature and effectiveness of the defense, the type and strength of weapon (bomb), the vulnerability of bases and factories. If such components can be assessed, so can c_1. But we must not forget that c_1 is complex only because the total picture is, and what we are working with is a preliminary simplification.

Of course, there are many ways to advance toward reality—and complexity. One, which we shall actually probe later in this chapter, is that in which it is assumed that the weapon depletion is proportional not only to the enemy's attack but also to the domestic supply. That is, the attrition destroys a certain fraction rather than a fixed amount of the target weapons. Thus the final term in (11.6.1) is replaced by $c_1\psi x_1 x_2$.

Another avenue is a three-way allocation, the new category being the assignment of certain weapons to the defense of "bases," that is, they offset the losses inflicted by the enemy. To write the kinematic equations in such a case, we would first have to analyze the subconflict between these offensive and defensive craft. The result would be certain expected losses to both sides and an expected penetration leading to diminished production as before. But it might require an analysis longer than the current one.

Of course, in any case we could take into account depletion of the weapons assigned to the various tasks. The simplest assumption, that of a fixed proportion, would adjoin to the expression for $\dot{x}_i$ terms like

$$-l_i\phi x_i - L_i(1 - \phi)x_i$$

where l_i, L_i are the average fractions of weapons lost that are allocated to the two purposes.

So far a homogeneity of weapons has been postulated. Reality would require several types (bombers do not bomb only bombers). Some could be, as up to now, used for the attritional sorties, defense, attack, or any combination; different types would suffer different restrictions. The number of state variables is now enlarged, the problem is more cumbersome, but there seems no reason why, with the requisite labor, it could not be solved.

Let us now turn to the payoff. It was supposed, in the early version, that there was a theatre of effective war detached from the attrition activity. In each time unit—for example, a day—each combatant assigns part of his weapons to this theatre, and they alone score in the payoff. The measure of merit for each side is the cumulative total—say the number of weapon-days—that he can deliver during a stipulated period of duration T. The payoff is the excess of these measures,

$$\int_0^T [(1 - \psi)x_2 - (1 - \phi)x_1]\, dt \tag{11.6.2}$$

so that each side strives to get a bigger weapon accumulation than his opponent.

Another payoff would be simply the excess of extant weapon at some given future time T. It would be terminal with $H = x_2 - x_1$. The desideratum here would be to store up the greatest possible excess in reserves of weapons over one's opponent; the effective war is in the future, rather than concurrent with the attrition.

One weakness is such payoffs is apparent at once; we have to know the duration T of the war beforehand. With what reliability can T be estimated and what is the penalty of error? Should a revision of this estimate occur during the conflict, the format of our theory makes the consequent best alteration of the optimal strategies easy and natural.[8] The consequences of a bad initial estimate will be softened thereby, and we should be able to assess how much by calculation.

Still another type of payoff assigns different values to the weapons depending on the time of their entry into the main war. (An airplane delivered early gives more service than a late one, say.) Such amounts to inserting a factor, say $F(t)$, into the integrand of (11.6.2). If $F(t)$ approaches zero sufficiently rapidly with increasing t, we can replace the upper limit by ∞. Here then is a payoff independent of any prior estimate of the duration T. It expresses the surplus of the total aircraft value; the price is the obligation to choose the valuation function F.

If we did not know T, but somehow felt we know a probability distribution for it, the payoff is of the form above with $F(t)$ being the probability density.

[8] For T itself is a state variable, and the optimal ϕ and ψ are always functions of such.

The most rational payoff would be based on a detailed analysis of the main war, generally a much harder problem than the present one. If we could express the Value of this war in terms of the delivery of distributions of weapons by the two antagonists, this functional would be the logical payoff for the present game.

But, among simpler payoffs, the imaginative reader will perceive many more possibilities, both of the genre we have described and others. The important point is not so much their realism or estimability, but whether the solutions—both the Value and the strategies—have traits essentially independent of the choice.

Let us now turn to the analysis we have actually done: Examples 5.4 and 11.9.1, the latter being placed later in this chapter to avoid interruption. In the former example we find essentially simple strategies. Let us recall that c_1 is a measure of the destructive power of E's weapons: one such weapon, when used for attrition, can cause a reduction of c_1 of P's weapons per unit time. For P, c_2 is similar and we have supposed $c_1 > c_2$. Then the optimal strategy for E, the player with the better weapons, is all-out attrition until a time $1/c_1$ short of the scheduled end of the conflict followed by an abrupt switch to all-out attack.

What does this mean? The $1/c_1$ can be interpreted, in an average sense, as the time required for a certain number of E's weapons to destroy a like number of P's. Thus we can attach a certain reasonableness to the criterion. When there is a greater time remaining than the above, E has a favorable exchange rate: his weapons have sufficient time to eliminate more than their number of the enemy's. As soon as this ceases to be so, it is advantageous to put all his forces into the direct war.

But P, with an inferior weapon, makes a similar switch earlier.[9] He waits until a time which precedes the scheduled termination by

$$\frac{1}{c_1}\sqrt{2c_1/c_2 - 1}.$$

Is there a heuristic interpretation here? This result seems to transcend, at least superficial, intuition. Of course, we should expect greater complication. For P, when making his decision, must anticipate the optimal behaviour of his opponent, which entails a strategy switch during the time remaining. On the other hand, when E calculated his best time to change, he could count on the later persistence of P's simple course of always fully attacking.

Note that these strategies do not depend on the manufacturing rates

[9] We are ignoring the curved part of $\mathscr{T}_2$ (Figure 5.4.2), where P switches earlier in anticipation of a weapon annihilation.

m_1, m_2 nor on the strength of the respective forces, x_1, x_2, but only on the efficacy of weapons. Such, of course, is not true of the Value.

By way of contrast, note the metamorphasis of the optimal strategies when we change the assumptions in the kinematic equations as we shall do in Example 11.9.1.[10] Effectively the innovation is that c_1 now means the fractional, rather than the absolute, rate of depletion of P's weapons by one of E. Similarly for c_2.

We find that, with certain limitations, E makes his transition from attrition to attack at time $1/c_1x_1$ before the end. This criterion is rather like the one before and admits the same interpretation. But now the transitions no longer occur consecutively, for P utilizes, under different circumstances (see Section 11.9 for the details), the analogous $T = 1/c_2x_2$.

But more interesting are the universal surfaces. Again, under certain conditions, E (and analogously for P), before he reaches the strategy transition, should endeavor to hold P's forces at the level

$$x_1 = \sqrt{m_1/c_1}. \tag{11.6.3}$$

He plays all attack or all attrition depending on whether x_1 is below or above this value. Once x_1 is there, E splits his efforts in the ratio

$$\psi = 1/x_2\sqrt{m_1/c_1}$$

to maintain the state (11.6.3). He does so until $T = 1/\sqrt{m_1c_1}$, whence he switches to all attack.

What does all this mean?

The next major step in a constructive program should be the study of the Value. How great are the penalties for deviations for the strategies just described?

11.7. THE BATTLE OF BUNKER HILL

The general setting is of two approaching antagonists firing at each other meanwhile. Each being constrained in the amount of available ammunition, the problem is to find the best distributions of their firepower, As the hit probability decreases with range, it increases during the approach. Firing too early then spells a probable ineffectiveness; firing too late allows the enemy too many unhindered shots.

Again we have a situation redolent of basic or classic military theory. The particular realizations are, of course, numerous. We can think of nearing ships or naval task forces, as interceptor closing in on an armed bomber, or, as our historic title suggests, of two bodies of infantry.

[10] The reader should glance over this example now to make the following paragraphs intelligible.

Analyses of such problems are not new. Duel games, in the simplest of which the two duelists approach each other, with a single shot pistol, date from the beginnings of game theory. The ideas have expanded from here to multishot and silent duels,[11] etc. and to cases of continuous fire, as the machine gun duel problem of John Danskin.[12]

We shall study from our present standpoint and partially solve such games where the antagonists are corps which become partially decimated by the enemy's fire.

One of the purest versions of such games has the kinematic equations:

$$\begin{aligned} \dot{x}_1 &= -x_2c_2p_2(T)\psi \\ \dot{x}_2 &= -x_1c_1p_1(T)\phi \\ \dot{m}_1 &= -c_1\phi \\ \dot{m}_2 &= -c_2\psi \\ \dot{T} &= -1, \qquad 0 \leqslant \phi, \psi \leqslant 1. \end{aligned}$$

Here x_1 and x_2 are the numbers of soldiers in two approaching armies, which are firing on each other meanwhile. The time until they will meet being T, the hit probabilities, p_1 and p_2, are decreasing functions of T. The quantities m_i are the (average) ammunition per man in the two armies and the c_i are maximal possible firing rates. Each side has the option of firing at any rate up to these; taking the fractions as the control variables gives us the third and fourth KE.

The total number of bullets fired per unit time by Army 2 (that is E, the minimizing player) is thus $x_2c_2\psi$. On the average we will suppose that the fraction $p_2(T)$ of these hit their mark; the total which do is the rate of decimation of Army 1. Such accounts for the first KE, the second following symmetrically.[13]

Observe the precise meaning of p_i. Its reciprocal is the average number of bullets needed to kill one enemy, a quantity which we suppose to decrease as the armies get closer.

The game terminates when the armies are close enough for the above approach phase to be no longer a valid description. With a suitable time

[11] See the account in Reference [10].

[12] Reference [11].

[13] The total bullets in Army i is x_im_i and the rate of change of this is $\dot{x}_im_i + x_i\dot{m}_i$. The first term is the effect on the firing rate due to the firers being decimated. We have neglected it, on the grounds that in a reasonable battle it should be small compared to the second term, the rate due to weapon use.

To take this effect into account is not hard. We form differential relations, as the above KE, and solve them algebraically for the $\dot{x}_i$, $\dot{m}_i$ to get our final KE. The results is more cumbersome in form than what we are using.

scale such occurs when $T = 0$, which is taken as defining $\mathscr{C}$. Of course, $\mathscr{E}$ is the set where all five state variables are nonnegative.

The best firing schedules (strategies) depend on what is chosen as payoff. Because the decision is typical of the quandaries facing the military analyst, let us look further at this instance.

One (cold-blooded) choice might be the difference in the number of surviving men. Such implies that we take

$$H = x_2 - x_1 \tag{11.7.1}$$

But note that our solution will reflect the choice. We should not be surprised if one or both sides arrive at the fray (after the approach) devoid of ammunition. For (11.7.1) puts the objective in terms of surviving men only; then, to optimize, each side will clearly want to use all its ammunition, if there is enough time (T) for it to do so.

To remedy the unreality of such an analysis, we might use as payoff the surviving excess firepower. That is, we take

$$H = m_2x_2 - m_1x_1. \tag{11.7.2}$$

If the surplus of men and the usable ammunition are both important, we could use for H a weighted linear combination of (11.7.1) and (11.7.2). But the weights (coefficients) are a result of a judgment, and we cannot expect the solution to be better than its validity.

Again, it might be that Army 1 is defending such a vitally important target from the invading Army 2, that the men of the former are expendable in comparison. Then both sides would be interested in the men or firepower that breached the defenses. Thus a suitable H would be x_2 or m_2x_2.

But there is only one strictly logical way to choose a payoff. We should consider the encounter which arises after the approach of the two armies and analyze it first as a separate problem. Its Value can be expected to depend on the x_i and m_i which were its inputs. Then this function of these four arguments should be the H of our original game.

Suppose now that H depended only on the x_i, such as (11.7.1). Observe that if the third and fourth KE were deleted, we would have a self-contained differential game with the state variables: x_i, x_2, T. Its solution would entail $\bar\phi = \bar\psi = 1$, for clearly the players do best by full firing at all times. But then our purpose, which is to find the best usage of a limited amount of ammunition, is lost. The role of the above two KE then is to supply these conservation constraints.

We shall partially solve two cases with payoffs (11.7.1) and (11.7.2). Both appear to have elaborately variegated solutions, and to avoid a gallimaufry of special cases we shall try to simplify. In both, we shall ignore

the bounds on the x_i. This allowing of negative numbers of men is not as absurd as it sounds. First, it is likely that in the cases of realistic interest, annihilation of neither side will occur; in fact, reasonable ranges of the variables will likely bound us well away from such a possibility. Secondly, the solutions obtained will be parts of the full solution. To extend them we need but add new components to $\mathscr{C}$ on which a requisite variable is zero. Let us, say, adjoin $\mathscr{C}_1$, defined by $x_1 = 0$ (and also $x_2 > 0$; other state variables $\geqslant 0$). The subsolution in $\mathscr{C}_1$ is trivial: the extinct Army 1 (P) cannot fire and E certainly will not; then $\bar\phi = \bar\psi = 0$, and to find V on $\mathscr{C}_1$ is trivial. We use it as an H with which to construct retrograde optimal paths emanating from $\mathscr{C}_1$ back into $\mathscr{E}$. The solution is completed by merging paths of this type with those of the unconstrained solution.

When we use the firepower payoff (11.7.2), we shall also ignore the positivity of the m_i. It has already been explained that such would be absurd for $H = x_2 - x_1$, but it is permissable in the present case on grounds like those of the two preceding paragraphs. It is not likely that a side will exhaust its bullets to attain a good payoff expressed in terms of them. If we wish to study such cases we can adjoin auxilliary $\mathscr{C}_i$ as above.

For reference, we write out the ME and RPE relative to the foregoing KE.

$$A_1\bar\phi + A_2\bar\psi - V_T = 0 \qquad (\mathrm{ME}_2)$$

where

$$A_1 = -c_1(x_1p_1V_2 + V_3)$$
$$A_2 = -c_2(x_2p_2V_1 + V_4)$$

and

$$\bar\phi = \begin{cases} 1 \text{ if } A_1 < 0 \\ 0 \text{ if } A_1 > 0 \end{cases}, \qquad \bar\psi = \begin{cases} 1 \text{ if } A_2 > 0 \\ 0 \text{ if } A_2 < 0. \end{cases}$$

while the RPE are

$$\begin{aligned} \mathring{x}_1 &= x_2c_2p_2\bar\psi & \mathring{V}_1 &= -c_1p_1V_2\bar\phi \\ \mathring{x}_2 &= x_1c_1p_1\bar\phi & \mathring{V}_2 &= -c_2p_2V_1\bar\psi \\ \mathring{m}_1 &= c_1\bar\phi & \mathring{V}_3 &= 0 \\ \mathring{m}_2 &= c_2\bar\psi & \mathring{V}_4 &= 0 \\ \mathring{T} &= 1 & \mathring{V}_T &= -c_1x_1p_1'V_2 - c_2x_2p_2'V_1. \end{aligned}$$

Example 11.7.1 The Battle of Bunker Hill: firepower payoff. We take as $\mathscr{C}$

$$x_1 = s_1, \quad m_1 = s_3, \quad T = 0$$
$$x_2 = s_2, \quad m_2 = s_4,$$

and, from (11.7.2),

$$H = s_2s_4 - s_1s_3.$$

Completing the initial conditions, as usual, by

$$V_i = \frac{\partial H}{\partial s_i}$$

we have, on $\mathscr{C}$,

$$V_1 = -s_3, \qquad V_3 = -s_1$$
$$V_2 = s_4, \qquad V_4 = s_2.$$

and so, still on $\mathscr{C}$,

$$A_1 = -c_1 s_1[s_4 p_1(0) - 1]$$
$$A_2 = c_2 s_2[-s_3 p_2(0) + 1].$$

There is full or no fire at termination depending on the signs of the brackets. Thus P, say, near termination, fires fully ($\bar{\phi} = 1$) when $s_4(= m_2) > 1/p_1(0)$ and not at all in the reverse case. When $\bar{\psi} = 1$, the test of Section 7.10 shows a void and there will be a universal surface.[14] We have not calculated it—a proper five-dimensional affair—but shall reflect on its significance.

It consists of those states at which P should exert such a partial rate of fire so that at termination $m_2 = 1/p_1$. This means that the enemy (E) soldiers each have (on the average) just the number of bullets which would give each P-soldier the capacity to kill (on the average) just one man.

An odd criterion? We should recall that this model depicts a sort of contest in the economy of bullets; each side expends them such as to have the maximal surplus in the end. The theory has furnished a criterion for doing so which does not appear obvious. (Of course, we recall that it embraces more states than just those of the universal surface; one side full fire, on the other none, until the US is reached.) Does it have a direct interpretation?

But the above type of action probably lies outside the range of practical interest. The one-man kill capacity implies that the approach phase was so long that it paid E to expend nearly all his ammunition during it; such would be expected to hold only in rather extreme cases.

The more realistic states, we should judge, will lie well on $\bar{\phi} = 1$ side of the universal surface. Then the partie concludes with full fire. Were this preceded by a no-fire phase, we should have a corroboration of Warren's command. That it is actually the case, under reasonable circumstances, will appear from Lemma 11.7.2, later in this section. For simpler analysis, we have assumed $p_1 = p_2$, and the remark that will follow the proof shows that the conclusion is a likely one in practice.

We have not computed the transition surface shown to exist. But, at least, when $p_1 = p_2$, this could be easily done with the aid of Lemma 11.7.1,

[14] It is not hard to show that if $\bar{\psi} = 0$, no ϕ — US can exist.

the preceding A_i, and the initial conditions. The surface would consist of those states when the "whites of their eyes" are visible.

Finally, it is likely that the full solution will embody a universal manifold for both players, that is, one to which both will steer. On it both control variables will assume intermediate values, say $\breve{\phi}$ and $\breve{\psi}$, and it can be expected to be of lower dimension (< 4) than a surface.

Because we have developed no general theory for such manifolds and have no grounds but instinct for asserting the existence of one in this game, the following must be regarded as spectulative.

Along a path on such a manifold, the intermediate control variables require that everywhere $A_1 = A_2 = 0$; then from the main equation, also, $V_T = 0$. The time derivatives of these three quantities also vanish. From $\mathring{A}_1 = 0$, we have, using the RPE,

$$c_2 p_1 p_2 \breve{\psi}(x_2 V_2 - x_1 V_1) - p_1'(x_2 V_2) = 0 \qquad (11.7.3)$$

and also $\mathring{V}_T = 0$ gives, directly from the RPE, for some λ,

$$V_1 = -\lambda c_1 x_1 p_1'$$
$$V_2 = \lambda c_2 x_2 p_2'.$$

Substituting these into (11.7.3) and solving gives us

$$\breve{\psi} = \frac{c_2 x_2^2 p_1'' p_2'}{c_2 p_1 p_2 (c_1 x_1^2 p_1' - c_2 x_2^2 p_2')}.$$

The analogous expression for $\breve{\phi}$ is obtained by exchanging the indices.

Writing $\breve{\phi}$ and $\breve{\psi}$ into the (left) RPE, we obtain a system of differential equations in the state variables. With a proper set of initial values, their integrals *may* give us our universal manifold.

Of course, final and more comprehensive statements require a more extensive analysis than we have undertaken here.

Example 11.7.2. The Battle of Bunker Hill: manpower payoff. The version with payoff being the excess of manpower at termination ($H = x_2 - x_1$) would apply to cases in which the events succeeding our battle of approach do not depend on the munition supplies.

Our treatment again shall be partial. We shall assume one side has enough bullets for continual fire, that is, $\bar{\psi} = 1$ always. Thus our problem will concern only the best distribution of the opponent's limited fire and so will be a one-player game.

This is not as drastic a limitation as it at first sounds. First, our limited problem is a good one in itself: how, for example, best distribute a small fusillade from heavy weapons against an approaching enemy who maintains a light but steady barrage at all times, the accuracy of fire increasing with time for both sides? Secondly, its solution covers most of the essential facets of the complete case. If the opponent always does not fire ($\bar{\psi} = 0$), the solution is trivial. Thus both steady possibilities of the opponent are covered. If the roles of the players are reversed, then the only features of importance omitted are possible universal manifolds of small dimension, where both players use intermediate values of their control variables and simultaneous transitions of them.

Our problem now has one control variable, ϕ. The state variable, m_2, can be suppressed; it plays no role. The KE are obtained from the previous set by placing $\psi = 1$ and dropping the fourth one:

$$\begin{aligned} \dot{x}_1 &= -x_2 c_2 p_2(T) \\ \dot{x}_2 &= -x_1 c_1 p_1(T)\phi \\ \dot{x}_3 = \dot{m}_1 &= -c_1\phi \\ \dot{T} &= -1. \end{aligned}$$

The ME is

$$A\bar{\phi} - c_2 x_2 p_2 V_1 - V_T = 0$$

where $A(= \text{old } A_1) = -c_1(x_1 p_1 V_2 + V_3)$ and $\bar{\phi} = \begin{cases} 1(A < 0) \\ 0(A > 0). \end{cases}$

The RPE are the same as before if we drop the fourth entry in each column and replace $\bar{\psi}$ by 1.

The universal surface for these KE has been computed in Example 7.9.2.

We will also need the readily computable result

$$\overset{\circ}{A} = -c_1 M \qquad (11.7.4)$$

where $$M = c_2 p_1 p_2(x_2 V_2 - x_1 V_1) + x_1 V_2 p_1'.$$

If P has enough ammunition to last until termination, his optimal strategy is obviously steady full fire: $\bar{\phi} = 1$. We are led to something like Example 7.14.1, in which the surface bounding such cases (see Figure 7.14.1) will probably act as a semiuniversal surface. Therefore we will use this surface as a terminal one and call it $\mathscr{C}_1$. It is characterized by there being just enough ammunition to last until $T = 0$; we parametrize it:

$\mathscr{C}_1$:
$$\begin{aligned} x_1 &= s_1 \\ x_2 &= s_2 \\ x_3 = m_1 &= c_1 s \text{ [15]} \\ T &= s. \end{aligned}$$

[15] Because of its prominent role, we write s for s_3.

There is another significant terminal surface. It may be in some cases optimal for P to do all his firing early and exhaust m_1 prior to termination. We encompass such a possibility by the terminal surface

$\mathscr{C}_2$:
$$\begin{aligned} x_1 &= s_1 \\ x_2 &= s_2 \\ m_1 &= 0 \\ T &= s. \end{aligned}$$

We shall treat $\mathscr{C}_2$ first. To ascertain H on it, we see how play will proceed from a typical one of its points. We obviously construe $m_1 = 0$ as implying a mandatory $\phi = 0$, and x_1 and x_2 will satisfy

$$\begin{aligned} \dot{x}_1 &= x_2 c_2 p_2(s - t) \\ \dot{x}_2 &= 0 \end{aligned} \qquad (11.7.5)$$

and $x_1 = s_i$ when $t = 0$. The $s - t$ appearing is T when it starts from the same point, (s_1, s_2, s) of $\mathscr{C}_2$. The sought H will be the value, as above, of $x_2 - x_1$ when $t = s$. The above system integrates to

$$\begin{aligned} x_1 &= s_1 - c_2 s_2 \int_{s-t}^{t} p_2(u)\, du \\ x_2 &= s_2 \end{aligned}$$

and so

$$H = s_2\left(1 + c_2 \int_0^s p_2(u)\, du\right) - s_1.$$

Our usual method then gives on $\mathscr{C}_2$

$$\begin{aligned} V_1 &= \partial H/\partial s_1 = -1 \\ V_2 &= 1 + c_2 \int_0^s p_2 \\ V_T &= s_2 c_2 p_2(s). \end{aligned}$$

Putting the initial conditions in the ME, it becomes

$$A\bar{\phi} = 0.$$

If there are to be paths entering $\mathscr{C}_2$, they must do so with $\bar{\phi} > 0$ and so $A = 0$ at such points, and consequently

$$V_3 = -s_1 p_1(s)\left[1 + c_2 \int_0^s p_2\right] < 0. \qquad (11.7.6)$$

These tributary paths into $\mathscr{C}_2$ can exist, as $A = 0$ there, only if $\overset{\circ}{A} < 0$ or if

$$M = M(s, s_1, s_2) > 0.$$

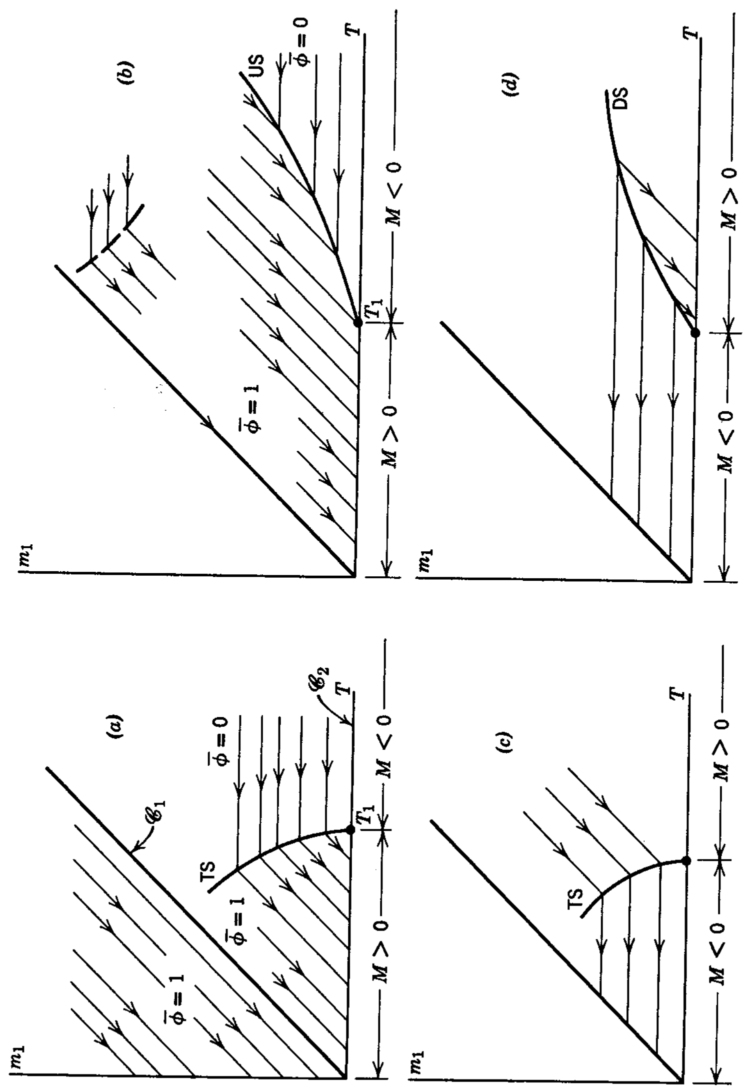


Figure 11.7.1

From (11.7.4), we find that

$$M = c_2 p_1(s) p_2(s) \left[s_2 \left(1 + c_2 \int_0^s p_2\right) + s_1 \right] + s_1 \left(1 + c_2 \int_0^s p_2\right) p_1'(s).$$

After evaluating this function on $\mathscr{C}_2$, we will have tributary paths when $M > 0$, that is, such will be states where P optimally concludes firing, and we should expect singular surfaces to emanate from the boundary where $M = 0$.

When $s = 0$

$$M = c_2 p_1(0) p_2(0) s_1 \left[\frac{s_2}{s_1} - J \right] \tag{11.7.7}$$

where

$$J = \frac{-p_1'(0)}{c_2 p_1(0) p_2(0)} - 1$$

and the solution depends rather critically on the sign of J (recall that $p_i'(0) < 0$, as the p_j are decreasing).

If $J < 0$, then, for small s (near termination), $M > 0$ and there will be cases where P should do all his firing early. If $J > 0$ such tributary path play occurs only when the force ratio s_2/s_1 is larger than J. Thus for a certain subset of cases (near $\mathscr{C}_2$ with s small and $J > 0$) P should fire early when his force is smaller than that of E and late when it is larger. To have $J > 0$ roughly means that P's firing accuracy falls off rapidly with range when the forces are close and c_2, E's rate of fire, is small. (We neglect the effect of $p_i(0)$ which should not be too far from 1.)

Due to the four-dimensional $\mathscr{E}$, we shall restrict our graphical depiction to the (m_1, T) plane. In such a diagram, the paths with $\bar{\phi} = 0$, where m_1 does not diminish, are horizontal, whereas if $\bar{\phi} = 1$, they have the fixed slope, $-c_1$. In Figure 11.7.1a, $\mathscr{C}_1$ (rather, a section of it) appears as a line of the latter slope through the origin; to the left is the region of surplus m_1, where $\bar{\phi} = 1$ always. A similar rendition of $\mathscr{C}_2$ lies on the T-axis, where $m_1 = 0$.

There should be a distinct such chart for each value of (s_1, s_2), but they are not cross sections of $\mathscr{E}$ for a fixed (x_1, x_2) because the latter changes value during a partie. Nevertheless, each play of the game will be accompanied by a moving point on such a chart which ultimately reaches $\mathscr{C}_1$ or $\mathscr{C}_2$.

At (a) and (b), $M > 0$ for small s ($= T$ when $m_1 = 0$). If for some s ($= T_1$), M changes sign, the diagrams suggest that the strategy switch might entail a transition or universal surface. Similarly, if $M < 0$ at small s_1 (c) and (d) indicate that a sign change generates a transition or dispersal surface. Thus it becomes important to study M as a function of s.

We shall not do so here meticulously. We shall suppose

$$p_1 = p_2 = p(\tau) = ae^{-k\tau}$$

with $0 < a \leqslant 1$, $k > 0$. No versimilitude is claimed for such probabilities; they merely fulfill our requirements ($p_i \leqslant 1$ and decreasing), yet lead to simple calculations.

Then

$$J = \frac{k}{ac_2} - 1$$

and

$$M = -p\left\{\left(\frac{s_2}{k}\right)(c_2p)^2 - \left[s_2\left(1 + \frac{c_2a}{k}\right) + 2s_1\right]c_2p + s_1k\left(1 + \frac{c_2a}{k}\right)\right\}.$$

The roots on the right are at $p = 0$ ($s = \infty$) and

$$p = \frac{s_2(1 + c_2a/k) + 2s_1 \pm \sqrt{(s_2 + c_2a/k)^2 + (2s_1)^2}}{2s_2c_2/k}.$$

Clearly, the latter two are positive. To correspond to a positive s, root $< a$. It is not hard to show that such cannot happen for the larger root, and for the smaller it happens if and only if

$$\frac{s_2}{s_1} > \left(\frac{k}{c_2a} - 1\right) = J.$$

Thus, if either

$$J \leqslant 0$$

or

$$\frac{s_2}{s_1} > J > 0$$

$M > 0$ for small s, but will change sign; one of (a) and (b) of Figure 11.7.1 will occur. To know which requires still further analysis, which will not be set down here. The criterion is the sign of $\mathring{M}$ at when $t = 0$, $s = T_1$ and such involves c_1, which has not appeared in the preceding criterion. Our own incomplete evaluation seems to indicate that either the transition or universal surface could appear.

In the event of the latter, if Lemma 11.7.2 applies, there must still be a transition surface for the $\bar{\phi} = 1$ paths; such as indicated by the broken curve at (b).

But if $s_2/s_1 < J$, $M < 0$ for all s. Then (c) and (d) cannot occur; the horizontal paths persevere rightward of $\mathscr{C}_1$. Here is pristine validity of Warren's command; P witholds fire completely until the last instant that permits its exhaustion.

The analysis for $\mathscr{C}_1$ parallels that for $\mathscr{C}_2$. In place of (11.7.5) we use

$$\dot{x}_1 = -x_2c_2p_2(s - t)$$
$$\dot{x}_2 = -x_1c_1p_1(s - t)$$

but from here on the procedure is the same. There is a new M, but it agrees with old when $s = 0$. Thus the criterion near the origin is just as it was.

There seems grounds for plausibly conjecturing (our analysis is too incomplete for certainty) that the conclusion of two paragraphs back holds generally:

Warren's command is valid if

$$\frac{x_2}{x_1} < \frac{-p_1'(0)}{c_2p_1(0)p_2(0)} - 1.$$

But, of course, our partial analysis indicates that the full solution will be quite complex. For instance, if there is a universal surface bordering on $\mathscr{C}_1$, it would entail firing programs of consecutive phases such as, first, a period of no or full fire (tributaries), second, partial fire (on the US in accord with (7.9.19), and third, full fire (on $\mathscr{C}_1$).

We conclude with the two lemmas cited several times.

When $p_1 = p_2 (= p)$ the RPE can be integrated in closed form under the full fire case when $\bar{\phi} = \bar{\psi} = 1$. The result is immediately extendable to the case where $p_1(T)$ and $p_2(T)$ have a constant ratio, but when they are essentially distinct, we are led to a form of Ricatti's equation.

LEMMA 11.7.1. The integrals of the system

$$\mathring{x}_1 = c_2x_2p(\tau)$$
$$\mathring{x}_2 = c_1x_1p(\tau)$$

with initial conditions: $x_i = s_i$ are

$$x_1 = \alpha_+Q + \alpha_-Q^{-1}$$
$$x_2 = \beta_+Q + \beta_-Q^{-1} \qquad (11.7.8)$$

where

$$Q = Q(\tau) = \exp\left(\sqrt{c_1c_2}\int_0^\tau p(u)\,du\right)$$

and

$$\alpha_\pm = \frac{1}{2}(s_1 \pm s_2\sqrt{c_2/c_1})$$

$$\beta_\pm = \frac{1}{2}(s_2 \pm s_1\sqrt{c_1/c_2}).$$

If the V_i, with the same RPE, have the values S_i when $J = 0$, they are also of the forms (11.7.8) but with

$$\alpha_{\pm} = \frac{1}{2}(S_1 \mp S_2\sqrt{c_1/c_2})$$

$$\beta_{\pm} = \frac{1}{2}(S_2 \mp S_1\sqrt{c_2/c_1}).$$

The proof is, of course, direct.

Suppose at some late stage in the game, $\bar{\phi} = \bar{\psi} = 1$. We want to know if there are transition surfaces so that earlier $\bar{\phi}$ or $\bar{\psi}$ was 0.

LEMMA 11.7.2. If, in the Bunker Hill game with $p_1 = p_2\ (= p)$, there is a set of points S at which $\bar{\phi} = \bar{\psi} = 1$ and $V_3 < 0$, then at some earlier time the paths through S will have met a transition surface preceding which $\bar{\phi} = 0$, provided that either

(1) $\int_0^\infty p(\tau)\, d\tau$ exists or

(2) $p(T) \sim \frac{a}{T}, \qquad T \to \infty$

and

$$2a\sqrt{c_1c_2} - 1 < 0. \tag{11.7.9}$$

Proof. The value of $\bar{\phi}$ depends on the sign of A_1, which by hypothesis must be negative at S. We are to show that it becomes positive for large τ.

From Lemma 11.7.1

$$A_1 = -c_1(p(\tau)[kQ^2 + O(1)] + S_3) \qquad (\tau \to \infty)$$

for some constants k and S_3, the latter being the value of V_3 at a point S, the KE informing us that V_3 is constant along a path. The form of the bracket is due to Q's being an increasing function of τ.

As $S_3 < 0$, if the term preceding it becomes small enough for large τ, our result is attained. Such happens under (1), for Q becomes constant and p, zero. Under (2) we are interested in, where k' and k'' are constants,

$$\frac{a}{\tau}\exp 2\sqrt{c_1c_2}(k' + a\log\tau) = k''\tau^{2a\sqrt{c_1c_2}-1}$$

so that (2) implies our result.

Let us note that, in three-dimensional physical space, the hit probabilities diminish, for large ranges, inversely as the square of the range.

If the range in our game decreases lineally with time, we will have

$$p(\tau) = O\left(\frac{1}{\tau^2}\right)$$

so that (1) will hold.

In the less likely case of two-dimensional aiming (2) will apply, and the validity of Warren's command would depend on the truth or falsity of (11.7.9).

Let us further note that the hypothesis $S_3 = V_3 = \partial V/\partial m_1 < 0$ can be counted on. Because Army 1 is minimizing, we should expect the Value to be smaller the more ammunition he has.

11.8. SOME PITFALLS IN ADAPTING GAME THEORY TO WARFARE

In the application of a mathematical theory to a practical science there are certain unavoidable limitations. Such general derelictions hold when we utilize game theory as a military tool as well as some peculiar to our subject. We list

1. An optimal strategy is, by its nature, the best possible, but it may be complex and the practical gains over a simpler, more obvious competitor slight. Such is a common phenomenon in many reaches of applied mathematics, but in game theory there is the innovation of an intelligent (at least so postulated) opponent. We must ascertain whether he can seriously exploit a deviation from an optimal strategy.
2. There are instances when we have assumed knowledge on the part of a player of certain parameters which he might not have in reality. (Note "parameters," a different affair from not knowing the opponent's state variables. The latter situation occurs properly in games of imperfect information and will be discussed in the next chapter.)
3. Let us recall the von Neumann definition of the solution of a zero-sum, two-player game. When both players utilize optimal strategies, the payoff will be the Value. But if one player plays nonoptimally, *there will exist a strategy* of his opponent enabling the latter to do better than the Value.

There are cases when the only such existing counterstrategies are non-optimal for the opponent. Then what is he to do? To exploit the defection of the first player, the opponent must play nonoptimally himself and so expose himself to a similar risk.

In differential games, fortunately, this phenomenon generally does not occur. If one combatant acts optimally and the other does not, the latter automatically suffers a penalty.

11.9. WAR OF ATTRITION AND ATTACK: SECOND VERSION[16]

In Section 5.4 we investigated a war game in which the control variables ϕ and ψ were the fraction of his weapons that a participant devoted to the destruction of his enemy's weapon production. His remaining weapons were devoted to the theater of war proper, and it is these alone that contribute to a favorable payoff.

The solution showed that if, say, P acted optimally, $\bar{\phi}$ was at first 1 (all weapons used for attrition) and then at some definite later time he switches to $\bar{\phi} = 0$ (attack with all weapons). No intermediate weapon allocations were ever employed. They are in the version below, revised slightly toward more complexity and better realism. As is generally true with linear vectograms, the agency is universal surfaces.

Our new KE will be

$$\dot{x}_1 = m_1 - c_1\psi x_1x_2$$
$$\dot{x}_2 = m_2 - c_2\phi x_1x_2, \qquad 0 \leqslant \phi, \psi \leqslant 1$$
$$\dot{T} = \dot{x}_3 = -1$$

with, as before,

$$G = (1 - \psi)x_2 - (1 - \phi)x_1.$$

As in the earlier version, ϕ and ψ are the proportions of their forces that the combatants devote to the long-range purpose of destroying the enemy's forces at their bases. The residual fractions, $1 - \phi$ and $1 - \psi$, enter the battle proper and contribute directly to the payoff. When ϕ or $\psi = 1$, we will thus speak of *attrition*; when it equals zero, of *attack*. The innovation here lies in the final terms of the first two KE and was discussed in Section 11.6.

As in the earlier version, $\mathscr{E}$ is

$$x_1 \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0$$

and $\mathscr{C}$ is

$$x_1 = s_1 \geqslant 0, \quad x_2 = s_2 \geqslant 0, \quad x_3 = 0.$$

Again it seems inadvisable, at least at first, not to bother with the maintenance of $x_1, x_2 \geqslant 0$. In fact, we shall not push the solution to exhaustion but shall be content with exhibiting its novel and important aspects.

The ME_2 is

$$m_1V_1 + m_2V_2 - V_3 + x_2 - x_1 - x_1S_1\bar{\phi} + x_2S_2\bar{\psi} = 0$$

[16] Likewise posed by A. Mengel.

where $S_i = c_2V_2x_2 - 1, \qquad S_2 = -c_1V_1x_1 - 1.$

Thus, clearly,

$$\bar{\phi} = \begin{cases} 1 & \text{if } S_1 > 0 \\ 0 & \text{if } S_1 < 0 \end{cases}, \qquad \bar{\psi} = \begin{cases} 1 & \text{if } S_2 > 0 \\ 0 & \text{if } S_2 < 0 \end{cases}.$$

Turning now to the initial conditions, on $\mathscr{C}$ we have $V_1 = V_2 = 0$ so that $S_1 = S_2 = -1$ and, as we should expect, $\bar{\phi} = \bar{\psi} = 0$, that is, all attack and no attrition late in the war. Near $\mathscr{C}$, the RPE are ($\mathring{V}_3$ proves superfluous)

$$\mathring{x}_1 = -m_1, \qquad \mathring{V}_1 = -1$$
$$\mathring{x}_2 = -m_2, \qquad \mathring{V}_2 = 1$$
$$\mathring{T} = 1$$

which have the integrals

$$x_1 = s_1 - m_1\tau, \qquad V_1 = -\tau$$
$$x_2 = s_2 - m_2\tau, \qquad V_2 = \tau.$$
$$T = \tau.$$

Thus

$$S_1 = c_2\tau(s_2 - m_2\tau) - 1 = c_2Tx_2 - 1$$
$$S_2 = c_1\tau(s_1 - m_1\tau) - 1 = c_1Tx_1 - 1. \qquad (11.9.1)$$

Thus transition surfaces occur on the hyperbolic cylinders obtained by equating the final terms of (11.9.1) to 0. Such surfaces, denoted by $\mathscr{T}_1$ and $\mathscr{T}_2$ have the appearance of Figure 11.9.1. The analysis thus far is

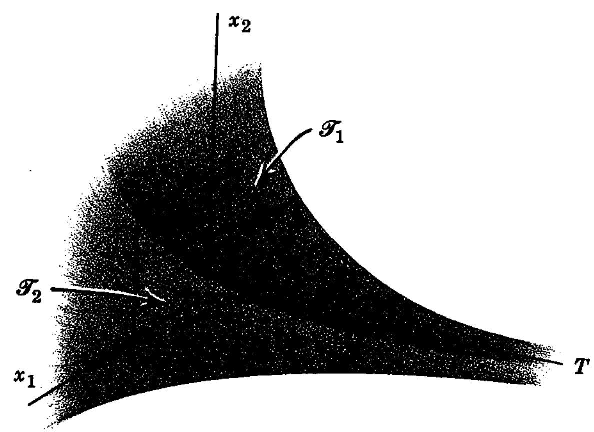


Figure 11.9.1

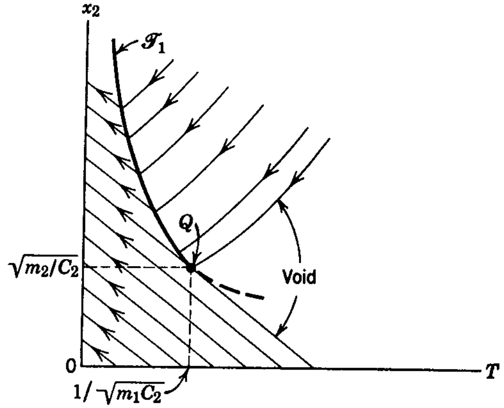


Figure 11.9.2

valid at most in the region behind their configuration as sketched (that the subset of $\mathscr{E}$ for which

$$Tx_2 \leqslant \frac{1}{c_2} \quad and \quad Tx_1 \leqslant \frac{1}{c_1}\Big).$$

Not all of this configuration will be significant. For let us envisage $\mathscr{E}$, for example, as projected on the (x_2, T)-plane (that is, a view of Figure 11.9.1 from along x_1-axis). The paths will appear as the oblique straight lines of Figure 11.9.2, and $\mathscr{T}_1$ as an hyperbola. Let it be tangent to the paths at the point Q. (Q is really the endwise projection of a line Q in $\mathscr{E}$.) Clearly only the part of $\mathscr{T}_1$ above Q will act as a transition surface because below Q, $\mathscr{T}_1$ will be met by none of the (retrograde) straight paths emanating from $\mathscr{C}$.

A brief calculation shows that Q has the coordinates

$$x_2 = \sqrt{m_2/c_2}, \qquad T = 1/\sqrt{m_2 c_2}. \tag{11.9.2}$$

We can expect (and in fact can verify) that crossing $\mathscr{T}_1$ leads to a change in $\bar{\phi}$ from 0 to 1, and so on the right side of $\mathscr{T}_1$ one of the RPE will be replaced by

$$\mathring{x}_2 = -m_2 + c_2 x_1 x_2.$$

Except at the boundary ($x_1 = 0$) the final term will be positive, and the paths emanating on the right of $\mathscr{T}_1$ will have greater slope than our former straight paths. Thus there will be a void just to the right of Q, and we act on this cue to seek a possible ϕ-US.

As on the straight paths, $\bar{\psi} = 0$, the universal surface will be sought on this premise. (Further, our criterion for a ϕ-US shows that none can exist

when $\psi = 1$.) The analysis was done in Example 7.9.3, and we found there the very appropriate candidate (7.9.23):

$$x_2 = \sqrt{m_2/c_2}.$$

Such is a plane through Q; of course, we utilize only the subset lying to the right of Q where

$$T \geqslant 1/\sqrt{m_2 c_2}.$$

To navigate on this plane let ϕ be $\breve{\phi}$. Then from the second of the KE

$$\dot{x}_2 = 0 = m_2 - c_2\breve{\phi} x_1 \sqrt{m_2/c_2}$$

or

$$\breve{\phi} = (1/x_1)\sqrt{m_2/c_2}. \tag{11.9.3}$$

As our constraints require $\breve{\phi} < 1$, the universal surface is confined to

$$x_1 > \sqrt{m_2/c_2}. \tag{11.9.4}$$

Our thinking will be simplified if we temporarily convert the situation to a one-player game by taking $\psi = 0$ always. In particular, we are disencumbered of $\mathscr{T}_2$. We are then dealing with a situation in which E attacks directly only, that is, he does not endeavor to destroy P's resources.

The universal surface appears as the quarter-plane BAC sketched in Figure 11.9.3. The tributary paths, as we already know generally, will merge smoothly with the family ($\bar{\phi} = 1$) emanating from $\mathscr{T}_1$ and with the family ($\bar{\phi} = 0$) coming directly from $\mathscr{C}$. The paths on the surface itself are sketched in the figure.

Let us interpret matters from our present state of knowledge. If termination is still far off (T large), and his forces not too small ((11.9.4) holds), P should *attack* ($\bar{\phi} = 0$) if his enemy's forces are below the level $\sqrt{m_2/c_2}$ and do so until they rise to this level. But if they are above, his strategy should be *attrition* until they are reduced to $\sqrt{m_2/c_2}$. When $x_2 = \sqrt{m_2/c_2}$, P holds it there by mixing attrition and attack in the proportion (11.9.3). He perseveres in this policy until the time $1/\sqrt{m_2 c_2}$ prior to termination, when he switches to full attack.

But against a large enemy force and with not too great a time to go, P's attrition policy will attain $\mathscr{T}_1$, and he switches to attack when this time $T = 1/c_2 x_2$.

These ideas are again illustrated by the cross-sectional diagram of Figure 11.9.4.

What happens near the hiatus bounded by QAC in the preceding figure? We shall show that there is a dispersal surface meeting it.

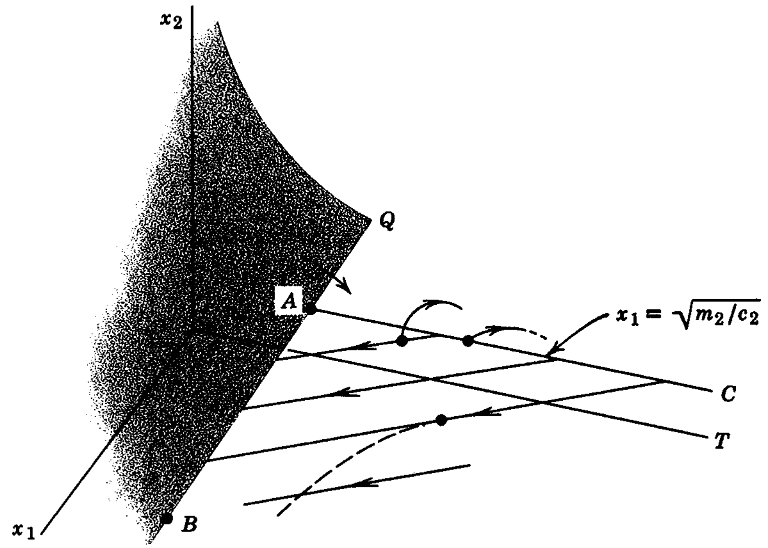


Figure 11.9.3

Relooking at the KE with $\bar{\phi} = 1$ (and $\bar{\psi} = 0$):

$$\mathring{x}_1 = -m_1 \quad (1)$$

$$\mathring{x}_2 = -m_2 + c_2 x_1 x_2 \quad (2)$$

$$\mathring{x}_3 = 1 \quad (3)$$

let us construct initial conditions on the universal surface near AC. For an

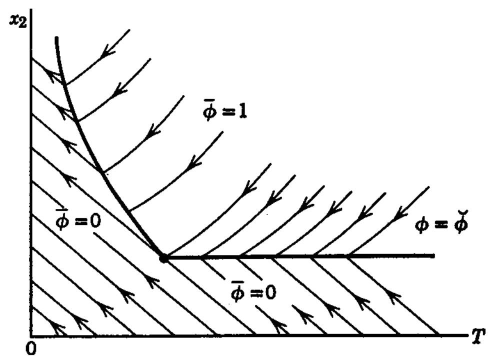


Figure 11.9.4

initial point on AC itself (2) shows that for $\tau = 0$

$$x_2 = \sqrt{m_2/c_2}, \qquad \mathring{x}_2 = -m_2 + c_2(\sqrt{m_2/c_2})\sqrt{m_2/c_2} = 0$$

$$\mathring{\mathring{x}}_2 = c_2(x_1\mathring{x}_2 + \mathring{x}_1 x_2) = -c_2 m_1 \sqrt{m_2/c_2} < 0$$

so that the paths, as they leave AC descend like τ^2. From starting points near AC, the paths will at first rise ($\mathring{x}_2 = -m_2 + c_2(\sqrt{m_2/c_2} + \varepsilon)\sqrt{m_2/c_2} =$ small and positive), then, by continuity, descend at this kind of rate. (One is sketched in Figure 11.9.3.) Thus they will meet the straight paths emanating from the lower side of the universal surface. From $\mathscr{T}_1$ near AQ, as x_1 is still smaller, the paths will descend even faster. Thus a dispersal surface can be constructed (Section 6.5) by equating the V on the two sets of intersecting paths.

Now let us restore the two-person game. First it will be convenient to know a little about V on the universal surface. We describe the latter by fresh parameters:

$$x_1 = s_1 \geqslant \sqrt{m_2/c_2}, \quad x_2 = \sqrt{m_2/c_2}, \quad T = s_3 \geqslant 1/\sqrt{m_2 c_2}.$$

Then we can show fairly routinely that on it

$$V_1 = -s_3, \quad V_2 = 1/\sqrt{m_2 c_2}, \quad V_T = 2\sqrt{m_2/c_2} - s_1 - m_1 s_3$$

and also

$$S_2 = -c_1 V_1 x_1 - 1 = c_1 s_1 s_3 - 1$$

and the surface is only acceptable in the two-person game at points where $S_2 \leqslant 0$ as its construction was premised on $\bar{\psi} = 0$. This means that we must discard all of it except the portion which lies behind $\mathscr{T}_2$, which has the equation $c_1 x_1 T = 1$. The boundary is indicated by the dashed curve in Figure 11.9.3.

We have no assurance that $\mathscr{T}_2$, aside from this dashed curve, will play its former role. What we must do is to evaluate S_2 along the tributary paths *from the residue of the universal surface* and ascertain the set of points on which it vanishes. Doing so for the paths *below* the surface is not difficult; we find the old $\mathscr{T}_2$ still in business. But the paths from above lead to differential equations recalcitrant to elementary methods, and we have not pushed matters to completion. Our conjecture is that S_2 becomes zero but on a surface *not* $\mathscr{T}_2$.

CHAPTER 12

Toward a Theory with Incomplete Information

12.1. INTRODUCTION

The value of extending our theory to games of incomplete information, for military and other applications, is clear. But the achievement is difficult and until recently little progress has been made. This chapter largely sows seeds rather than reaps results. It portrays the difficulties, conjectures what solutions would be like and seeks to delineate the most promising ways of attaining them.

In the following section is the first glimpse of the inevitable introduction of mixed strategies. As in the standard theory of the zero-sum, two-player game, such consist of a player's decisions being made in accord with a definite probability distribution. Just how this should be done in differential games is not obvious. The most likely possibilities are suggested by realistic prototypes of the games. After a tentative general definition of a differential game with incomplete information, we turn to specific types using as models some actual instances of firing and pursuit games. The search game, as one with minimal information is termed, often appears likely to furnish a constituent of the solution of more general cases.

Search games constitute Sections 12.3 and 12.4. In the first, we prove that when the hidden objects are numerous and immobile, the time to find them (the payoff) is nearly independent of the searcher's strategy as long as no effort is wasted researching territory already scouted. This striking result, as far as we know, is the only one definitely establishing the practical validity of approximate methods, an approach we feel has great possibilities. This cause is argued in some detail in Section 12.6.

Section 12.4, on search games with mobile hiders, again takes us to an unexplored realm. We conjecture here that the details of the randomization are unimportant, but certain basic parameters, such as the hider's speed, are not. Again there appear to be grounds for an approximate theory.

For certain games with a "stationary" or "steady-state" character there appears to exist a technique for solution, albeit a tedious one (which might be alleviated by approximation). It is expounded in Section 12.6 and exemplified by a firing and evasion game.

12.2. A SPECULATIVE PURVIEW

One of the main difficulties of differential games with incomplete information is that undoubtedly optimal play will, in all essential cases, require mixed strategies. What form will they assume in a differential game? A mixed strategy customarily means the randomizing of a player's decisions in accordance with some probabilistic law. Just how is a player to randomize his choice of control variables, quantities which he controls continuously?

Whatever the nature of the strategies, there is a basic distinction from games with full information. Generally strategies can no longer rest on the concept of the control variables being functions of the state variables, that is, a player's decision depending on the current position only. With partial information his knowledge of the *past* states will generally motivate his current decision.

The form of the information enjoyed by a player will be a probability distribution over the state variables at each instant. As full knowledge is a particular such distribution, the games with complete information, discussed heretofore, are special cases.

There are two sources of this information. Some is *granted*, that is, furnished[1] by rules of the game which may be formulated for this purpose. For example, if the game were modeled on a physical situation in which one player obtained knowledge of his rival through an imperfect detection device, the rules might be designed to simulate the imperfection by supplying him with equivalent granted information. Each player must also be granted some data as to the starting position; the most general form is a probability distribution over $\mathscr{E}$.

There is also *inferred* information. A player will know the full history of his own control variables[2] and he can infer from it more than the granted information.

[1] The case where no information is furnished is not excluded.

[2] Such is usually the case, but there are exceptions. For example, if one "player" is a team whose members are in imperfect communication.

Example 12.2.1. Let the kinematic equation be

$$\dot{x} = 2 + \phi$$
$$\dot{y} = 2 + \psi, \qquad -1 \leqslant \phi, \psi \leqslant 1.$$

Here P, controlling ϕ, will always know the current value of x if he knew its starting value x^0 (by integration of the first KE); such is inferred information. If he had only a probability distribution of x^0, at any later time his knowledge would be this same distribution translated by a fixed amount, which is always known to him via $\phi(t)$ through integration.

But if he knew y_0, the starting y, all he could tell of y at time t would be

$$y_0 + t \leqslant y \leqslant y_0 + 3t.$$

But there may be further rules of the game that supply him with a finer knowledge; such would be granted information.

Of course, it is quite possible to have optimal pure strategies in cases of only partial information. But in many of the more significant examples, based to some extent on reality, it is intuitively clear that mixed strategies are demanded.

Research Problem 12.2.1. Let us suppose a game of incomplete information with one control variable apiece for the players and linear vectograms. In the solution of the corresponding game with full information, $\bar{\phi}$ and $\bar{\psi}$ are to assume their extreme values almost everywhere in $\mathscr{E}$; let us say $\mathscr{E}_{\pm}$ are the subsets of $\mathscr{E}$, where $\bar{\phi} = \pm 1$. Then, returning to a partie of the original game, at some instant P will have available a probability distribution, granted, inferred, or composite, for $\mathbf{x}$ over $\mathscr{E}$; he can then always compute the probability that $\mathbf{x}$ is in $\mathscr{E}_{+}$ or $\mathscr{E}_{-}$. A plausible strategy is to play, according to which is greater, $\phi = +1$ or -1. Under what conditions is this strategy optimal? (One might use Example 12.2.1, with various $\mathscr{C}$, G and H to experiment along these lines.)

For less abstract examples, let us consider pursuit games. As usual P pursues E, E evades P, but each (or possibly only one) has incomplete knowledge of the other's position. Reality, especially the limitations of actual sensing devices, suggests a number of forms for this incompleteness (for definiteness we shall speak to P's granted knowledge of E, but of course all possibilities may hold reciprocally):

1. P may know E's location (rather) well but may have little information as to his other state variables (if there are any), such as heading and speed.

2. P's sensing device gives him only a probability distribution of E's position.

3. P may know E's relative bearing only, that is, the inclination of line PE.

4. There may be a time lag; P can act only on E's position some fixed time T ago.

5. P receives his information intermittently. (Suggested, of course, by sweeping radar, but there is game-theoretical interest only if the interim between signals is great enough to allow E to maneuver significantly therein.)

When no information[3] is granted to either player, we will speak of a *search game.*

We will discuss some possibilities indicated by some of the five categories.[4] Let us take a simple cookie-cutter version of **2**, that is, at each instant the only information granted P as to E's whereabouts[5] is that E is equiprobably within a sphere of radius r. In a discrete model, where the players alternate jump type moves, we might prefer to think of a chance move inserted preceding each one of P. It would assign him his detection sphere by choosing its center equiprobably in a sphere of centre E and radius r.

For the moment adhering to the discrete version, the information that P receives may be as indicated in Figure 12.2.1a. The dots are successive positions of E, but P only knows that each is equiprobably within its containing circle.

Suppose there comes a time when two successive circles appear as at (b),

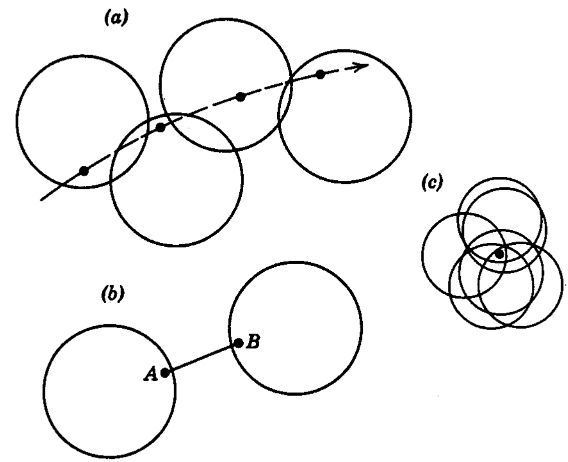


Figure 12.2.1

[3] Except possibly some relating to the starting position.
[4] Our illustrations do not suffer if we leave E's information about P unspecified.
[5] In a space whose coordinates are all the state variable pertaining to E (his position and possibly his direction, speed, etc.).

where AB is only very slightly less than the span available to E in his last move. Then P knows that E must be very near B. This phenomenon need not be uncommon. The special case where E is stationary will, in time, result in something like (c) of the figure; if P waits long enough, with probability one, he can infer as exact information as he pleases.

Thus the inferences of accumulated granted information may be much greater than the current allotment. This type of phenomenon again explains why a good strategy must depend on past states.

The opulent overlapping of spheres, as at (c), is in part due to our stipulation that they are probabilistically independent. But what happens if we take finer quantizations in an effort to approach a continuous game? We would come ever nearer to certain determination with ever shorter intervals, ultimately tantamount to no error at all, to a game with full information.

A remedy for such an absurdity emerges if we look at actual continuous measuring instruments and the science of their errors, a part of the theory of stochastic processes. Common in this subject are autocorrelated functions which have small probabilities of large changes for small increments of their arguments. Obviously a constraint of this type is needed for realism in the continuous game.

There is an analagous unreality in strategies. It seems clear the optimal play will be mixed in games with essential limits on information. Yet how is the randomization of a control variable to be effected? At each instant, to allow a player the choice of a value from an independent probability distribution smacks of an absurdity similar to the above. For any realistic model, the continuous choice, say, of a rudder position demands that near successive positions be correlated whether the agent be man or mechanism. Again, it seems, we must speak with a stochastic accent.

U. Grenander, in a splendid work of eighty-four pages,[6] has shown the way. He deals with pursuit games which have a steady state character, and the decisions of the players are proscribed by stochastic means such as appear in prediction theory.

Let us return to the pursuit game and spectulate as to what the full solution may be like. To illustrate possibilities we will make the artificial assumption that r, the detection sphere radius, is constant[7] and large in comparison with the capture radius.

If the partie begins with the distance PE much greater than r, we can expect the early play to be like that of the game with full information; P will pursue the relatively small sphere as he would E and E will employ corresponding evasive tactics. But with proximity, especially after P enters

[6] Reference [12].

[7] Instead of diminishing with proximity as would be likely in practice.

the sphere, the lack of information will introduce a searching and hiding phase. If we suppose a very high autocorrelation so that the detection sphere is virtually stationary and that, once P gets within it, he so remains without much sacrifice of agility, the ensuing play will be virtually a search game with the interior of the sphere acting as $\mathscr{E}$.[8]

This pure search aspect fits our pursuit game only if P has such great kinematic advantage over E that the latter can be regarded as relatively stationary.[9] But in other, and probably more general, cases P will be obliged to divert some of his effort to keep up with the moving sphere. Thus his strategy will be a mixture of pursuing E (say, by pursuing the center of the sphere) and the random search as above. Similarly, E must blend an evading strategy with his random hiding one.

In any case, the search game will be a constituent, and we will discuss such games in the next two sections.

Is the transition between the phases—the early pure pursuit and evasion and the later one with a blending (or entirety) of mixed searching and hiding—gradual or abrupt?

Now let us ruminate on some of the other five types of partial information mentioned earlier in this section.

If **3** holds, so that P only knows E's relative bearing, again the knowledge of E's position will accumulate with the passage of time. Were E stationary, P, with sightings from only two positions, could spot E precisely by triangulation. For mobile E, P will have a harder task, but how much so will depend on his knowledge of E's kinematic limitations, especially his speed.

We can readily envisage strategies of E deliberately designed to frustrate P's garnering of information. They will have to be random, of course, for (once again) if P had a forecast of E's locations he could triangulate on them as well as against current positions.

Under **5**, with P's sightings intermittent, periodic, and widely spaced, we can muse on the advisability of E's mixing over courses equiperiodic with the P's sightings as in Figure 12.2.2. The deception ensues from P's spotting E only at the points marked by large dots.

Situations with a time lag **4** are commonplace. For example, feinting is

[8] At this point we can convince ourselves of the need for random strategies. In a search game P and E within a sphere $\mathscr{E}$, suppose P had an optimal pure strategy. It would direct him to search the portions of $\mathscr{E}$ in a certain order. But E being able, too, to compute this strategy, need merely always stay in a part of $\mathscr{E}$ remote from P. Similarly a pure optimal strategy for E would supply P with an advance schedule of E's positions. Each player can only outwit the other through "mixing"—picking one of a variety of routes through chance.

[9] That is, if we take coordinates centered at E, the motion of the sphere hinders P but slightly and he is free to search it.

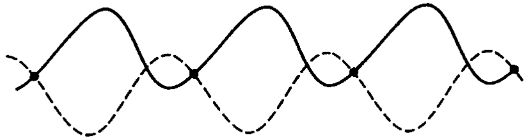


Figure 12.2.2

effective only when there is a lag between the opponent's observation and his active response. Let us reconsider the earlier football examples with a ball carrier E confronted by a tackler P. A feint by E, such as a lunge to the left, followed by a quick veer around P's right, would be futile in its attempt to throw P off balance if the latter were capable of instantaneously reacting to E's movements.

We have discussed a number of ways the players might move for the sake of information—P to gain it, E to deter it—which almost always entail randomizing and have suggested that, in pursuit games, optimal strategies should be a blend of these type moves and those of more direct pursuit and evasion, resembling (at least) the optimal strategies were the information complete. Now it is not certain that such will always be the case. The former type of moves—those concerning information—generally penalize the latter—those of direct pursuit and evasion. There will be instances when the penalties are too high—the example below is an extreme case—and another basic facet of the general problem is to find criteria.

Example 12.2.2. The simple pursuit game. We revert to Example 1.9.1 in which P and E move in the plane, each with simple motion, P having the greater speed. The payoff being time until capture, we know that optimal play consists of P's chasing E along the straight line through their starting positions. Now suppose P has only information of type **3**; he only knows the relative bearing angle of E, not the range. But the bearing angle is all he need know to play his optimal strategy of the game with full information. Obviously this strategy is best here, too. Neither player should do any mixing.

It would be possibly advisable for P to veer off course to triangulate only if the game were altered in such a way that the extra information were advantageous to him. We leave it to the imaginative reader to devise such alterations.

Similar, but less extreme, ideas apply to our other cases. For example, the oscillatory routes of Figure 12.2.2 will curtail E's effective evasion speed. With a payoff such as capture time, it is doubtful whether he should employ them.

But if we shift our thinking from pursuit to firing type games, the situation changes. Here P is equipped with a weapon such as a gun (torpedo, missile), and he may fire one or a succession of shots at E. Let us first ignore any inherent inaccuracy in the weapon, so that P is certain to score a hit if he knows E's location exactly. The payoff is the hit probability.

Then, as E's only objective is to degrade P's information, it is clear that randomized strategies are essential. Similarly, P will have to mix, for any set policy of when and where to fire would enable E to be not at target location at the time.

A generic instance of such games of great practical importance illustrative of these ideas is embodied in

Example 12.2.3. The problem of delayed firing and evasion. An instance of **4**, there is a time lag between P's sighting of E and the arrival of the projectile. Let us say P is allowed one shot, which he must aim at some future location of E. The objective of E, assumed mobile, is to maneuver so as best to confound P's prediction. Randomization is the essence, for any systematic zigzag is as predictable as uniform motion. But just how is a difficult matter, for we assume that E knows nothing of the shot until the arrival of the projectile and P is free to fire at any time. It might appear that at any instant he minimizes P's information with a mixed strategy that results in equiprobable distribution of E's location over all possible places. But not knowing when the blast will come, he ought be equiprobable at all times and this is impossible.[10]

Of course, one can coin many explicit games of this ilk. One such, the simplest nontrivial one possible, will reappear as Example 12.6.1.

Pure aiming and evasion games, such as the previous example, which have a sort of "stationary" character (see Section 12.6), and the objectives can be expressed in terms of information alone, are grist for the theory of stationary stochastic processes. The extant techniques of optimal prediction can be utilized for P and reversed for E. That is, the latter player seeks that haphazard course which maximizes the prognostic error of his whereabouts. Grenander's reseaches in this direction are excellent and deep and very likely point to a future theory that is complete, elegant, and useful.

But once stationarity is waived, our ideas cloud again even when the payoff embodies prediction alone.

For example, let us waive our assumption of the absolute accuracy of P's firing and suppose that it diminishes with range, the distance of E away. Then it seems that E's random strategy must incorporate, to some

[10] For an extended explanation, see the fourth of References [13].

extent, fleeing from the weapon site. To what extent? Can the rate of decrease of accuracy with range become so critically great that, when exceeded, E's optimal strategy is the pure one of just fleeing?

Another variant is had by adding a destination for E; once there, his mission is accomplished; P's objective is a prior hit. Let us so modify Example 12.2.3 and also suppose P to have more than one bullet. When E is fairly close to his target, going there directly makes his location perfectly predictable. Yet using random motions to disguise it keeps him vulnerable to fire longer. Again the dilemma of fusing a pure and mixed strategy!

Another instance is a simplified version of a case of vital importance:

Example 12.2.4. Invader interception following early warning. An enemy bomber (or guided missile) E is detected while still far from a known target. Its speed and heading as well as its location are known at some instant, as might be the case with a DEW (Distant Early Waring) line. Defending interceptors instantly are launched; one is shown starting from P in Figure 12.2.3, W being the point of detection. A naive defensive strategy would be the straight collision course based on the assumption that E persists in a constant velocity. The dashed paths in the figure show the ensuing capture occurring when P is at C which is surrounded by a circular capture region.

But what if E takes a less direct route to the target? The solid paths sketched suggest some possibilities (for emphasis, let us not fear fantastic deviations).[11] If E randomizes over some set of these, the interceptors

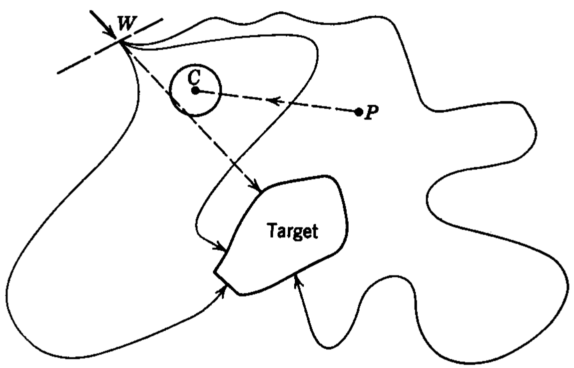


Figure 12.2.3

[11] We suppose no fuel limitation on E.

are faced with what is essentially a search game. Their object is to discover E before he reaches the target.

We cannot solve this game whose optimal strategies are certain to be mixed. But it is another illustration of the blending quandary. Use of extreme, outré paths by E will make finding him difficult, but their great length leaves him vulnerable too long and so deters his probability of success. How best should he compromise between the pure—going directly to target—and the mixed strategies?

12.3. SEARCH GAMES WITH IMMOBILE HIDERS

In the simplest case one player E, the "hider," secretes an object somewhere in a region $\mathscr{R}$, which may be in a space of any dimension number. His opponent P, the seeker, strives to find it in the least time. As E endeavors to maximize this time of search, it is the payoff.

We shall suppose simple motion on adequate approximation to P's mode of travel—a fixed speed with complete freedom in his choice of direction. Surrounding P will be a fixed region of surveillance, let us say a sphere of diameter d. The object is found when it is contacted by this region. Then, as P searches, he cuts a swath of diameter d in $\mathscr{R}$. A traverse of $\mathscr{R}$ by P in which all of $\mathscr{R}$ is searched without duplication will be called a *tour*.

We are going to be tolerant in this definition. For example, with a circular surveillance region, P could not execute a perfect tour of a square $\mathscr{R}$. But we shall overlook small transgressions, such as the overlaps that occur on sharp bends of P's path or small protrusions outside of $\mathscr{R}$ due to serrations or sharp corners in its boundary. Thus we shall suppose that the cross-section "area" of P's swath times the length of a tour is the "volume" of $\mathscr{R}$. (The words in quotes apply to a three-dimensional $\mathscr{R}$; for a planar one read "width" and "area"). Consequently, the length of tour is fixed, and, as P moves at a given speed, so is the time required. The latter we will call T. In practice, a sensible complete search of $\mathscr{R}$ without unavoidable duplication or waste should require but little more time than T and we neglect this excess.

In this *simple search game* E has one move: he places the object anywhere in $\mathscr{R}$. Then P endeavors to find it as quickly as possible, starting from any point of $\mathscr{R}$ he wishes.

THEOREM 12.3.1. The Value of the simple search game is $\frac{1}{2}T$. The only optimal (mixed) strategy of E is to place the object equiprobably in $\mathscr{R}$. *An* optimal strategy for P is to traverse some tour and its reverse (the same route traveled in the opposite direction), each with probability $\frac{1}{2}$.

Proof. Let us think of the route of some tour unfurled into a straight line. Then, to within a reasonably small error, each hiding place E might select in $\mathscr{R}$ can be identified with the point of the tour (line) at which the object is discovered.

1. Let E play the equiprobable strategy. If P searches via any tour, then the object's location will have a uniform probability distribution over the line. As P traverses this line with constant speed, the expected time to discovery is one-half the total or $\frac{1}{2}T$. If P elects any search scheme not a tour, it is clear that its inefficiency will render the expected time $\geqslant \frac{1}{2}T$.

2. Let P play the strategy of the theorem. A pure strategy by E is tantamount to his selection of a point on the unfurled tour. If U is the resulting time of discovery for a traverse in one direction, this time will be $T - U$ for the other direction. The payoff is then $\frac{1}{2}[U + (T - U)] = \frac{1}{2}T$. The theorem follows from the standard definitions of game theory.

Remark. We see from the proof that the liberty granted P of choosing his starting point is far more generous than necessary.

Problem 12.3.1. Give an optimal strategy for P essentially distinct from that of the theorem or a mixture of such for different tours.

If P governs a team of s identical searchers, it is evident that he should divide $\mathscr{R}$ into s portions of equal area and assign one searcher to each. Optimal play for P results when each searcher acts optimally in his portion. Thus we have

COROLLARY 12.3.1. The Value of the simple search game with s searchers (and one hider) is $\frac{1}{2}T/s$.

On the other hand, if the number of hidden objects is augmented, the payoff being the time required to find them all, the difficulty of the problem is vastly increased. Ascertaining the optimal strategies is a recondite business and they can depend on the shape of $\mathscr{R}$. For example, if $\mathscr{R}$ were long and narrow, with cross-section diameter $\leqslant d$, the only tours possible are the two which go from one end of $\mathscr{R}$ to the other. With one searcher and two hiders, E's best strategy is to place the latter at the extremities of $\mathscr{R}$ for the payoff will then be at least T, which is the best possible. For a spherical $\mathscr{R}$, on the contrary, it is evident that no pure strategy can be optimal for E. Even more difficult are optimal search strategies.

But—and this our main point—if the number h of hiders is at all large, the searching strategy matters little as long as it entails no duplication. That is, P may allocate to his s searchers each a tour over $1/s$th of $\mathscr{R}$. Even if he announces the full strategy and E exploits this knowledge by hiding in the last places the searchers will look, the payoff will not be greatly increased over the Value.

More precisely, we shall essentially prove

$$\frac{h}{h+1}\frac{T}{s} \leqslant V \leqslant \frac{T}{s} \tag{12.3.1}$$

where V is the Value. The inequality shows that for h reasonably large, say 10, V is not far from T/s, the duration of a cooperative tour. Thus *any* such tour will net P a payoff close to the Value.

We shall derive the counterpart of (12.3.1) for a discrete model. Let $\mathscr{R}$ be peppered with dots, contiguous ones being connected by lines; thus it is approximated by a linear graph. The new version of our game is the obvious one: E secretly places the h hiders on any distinct dots; P in turn starts the searchers at any s points and at each move shifts each to an adjacent (line connected) point. A hider is found when a searcher occupies its point. The payoff is the number of moves which finds them all.

We shall take it that a sufficiently fine and regular graph simulates the continuous game closely. It is convenient and harmless to suppose that the number N of points is divisible by s.

The result on discrete games, Theorem 12.3.2 below, will be taken as adequate evidence for (12.3.1).

Let v be the payoff when the strategies are as follows: E plays equiprobably, that is, the hiders are distributed randomly with all subsets of h points equally likely; P plays any definite multiple tour, that is, no searcher ever moves to a point previously searched.

LEMMA 12.3.1

$$v > \frac{h}{h+1}\frac{N}{s}.$$

Proof. If all hiders are found on the kth move, P will have covered ks points. The h hiders must have been at these points with at least one occupied that was explored on P's final move. The number of ways of placing h hiders on ks points is $\binom{ks}{h}$. We deduct the unwanted cases, where all hiders are on the $(k-1)s$ nonfinal points, which number $\binom{(k-1)s}{h}$. The difference, when divided by $\binom{N}{h}$, the total number of hider allocations, is the probability that the payoff is k. As v is the expectation of the payoff

$$v = \sum_{k=0}^{M} k \left[\frac{\binom{ks}{h} - \binom{(k-1)s}{h}}{\binom{N}{h}}\right]$$

where $M = N/s$.

By rearranging ("summation by parts"),

$$v = \frac{M\binom{Ms}{h} - \sigma}{\binom{N}{h}}$$

where

$$\sigma = \sum_{j=0}^{M-1} \binom{js}{h}.$$

As $Ms = N$,

$$v = \frac{N}{s} - \frac{\sigma}{\binom{N}{h}}.$$

To assess σ we utilize the familiar relation

$$\sum_{i=0}^{n} \binom{i}{h} = \binom{n+1}{h+1}.$$

Now

$$s\binom{sj}{h} \leqslant \binom{sj}{h} + \binom{sj+1}{h} + \cdots + \binom{s(j+1)-1}{h}$$

which is obvious if the left side is regarded as a sum of s equal terms. If we sum over j and note that "numerators" on the right are totally the consecutive integers from 0 to $sM - 1 = N - 1$, we have

$$s\sigma \leqslant \sum_{i=0}^{N-1} \binom{i}{h} = \binom{N}{h+1}.$$

Finally,

$$v \geqslant \frac{N}{s} - \frac{\frac{1}{s}\binom{N}{h+1}}{\binom{N}{h}} = \frac{1}{s}\left[N - \frac{N-h}{h+1}\right]$$

$$= \frac{h}{h+1}\,\frac{N+1}{s} > \frac{h}{h+1}\,\frac{N}{s}.$$

THEOREM 12.3.2. For a discrete version with N points of the simple search game with h hiders and s searchers, the Value V satisfies

$$\frac{h}{h+1}\,\frac{N}{s} < V \leqslant \frac{N}{s}.$$

Proof. As N/s is the number of moves in a multiple tour, which certainly must discover all hiders, the right inequality is clear.

Let E play equiprobably. Any tour by P leads, by the lemma, to a payoff $> (h/h + 1)(N/s)$. This inequality applies for all strategies of P, for any such with duplications could be replaced by a better one without them. Because E is the maximizing player and has a strategy guaranteeing a payoff satisfying this inequality, V must satisfy it too.

Research Problem 12.3.1. Analyze games like the foregoing except that they terminate when a certain number ($< h$) or fraction (< 1) of the hiders is found.

12.4. SEARCH GAMES WITH MOBILE HIDERS

As far as we know, virtually nothing is known about the solution of such games. The following instance appears to embody the quintessence of the problem.

Example 12.4.1. The princess and the monster. The monster P searches for the princess E, the time required being the payoff. They are both in a totally dark room $\mathscr{R}$ (of any shape), but they are each cognizant of its boundary (possibly through small light-admitting perforations high in the walls). Capture means the distance $PE \leqslant l$, a quantity small in comparison with the dimension of $\mathscr{R}$. The monster, supposed highly intelligent, moves with simple motion at a *known* speed w. We permit the princess full freedom of locomotion.

We do not know how to solve this problem, but it seems certain that the optimal strategies will be highly randomized. Our feeling is that just how the players use chance to pick their *paths* is secondary; probably any haphazard meandering will do about as well as any other.

We conjecture that the sole decision of importance rests with E. How fast should she run? One extreme—complete immobility—seems unpromising. For P, by a tour of $\mathscr{R}$, can be certain of a payoff not exceeding the fixed duration of such, whereas any sort of motion by E offers at least the possibility of an arbitrarily long period of freedom.[12] At the other extreme, a very high speed (compared to w) by E can hardly be desirable, for in a short while capture is nearly certain; she will run into the monster.

Somewhere between the extremes, then, will be an optimal speed for E. Where?

This example typifies the problem of pure search games. The following one is similarly unsolved but appears simpler and may possibly serve as a stepping stone.

[12] Of course, it may be that the *expected* period is less than the time for a tour, but it seems unlikely.

Example 12.4.2. A simpler princess-monster game. The only innovation is that now P and E are each confined to a closed curve. We take it as a circle.

One simplification is that at each instant, if their speeds are assumed fixed, the players each have but two directional choices.

The conjecture as to E's optimal speed now appears to have a natural candidate: E should employ w, P's speed. Our shaky grounds are merely that equal speeds alone preserve the distance PE should both players use the same pure strategy.

Research Problem 12.4.1. Solve the discrete version of this game. The players each occupy one of n ($\geqslant 3$) points distributed on the circumference of a circle. They move alternately, transferring their positions to either of the two adjacent points. Capture occurs when either both players occupy the same point or they occupy adjacent points and interchange positions after one move apiece. The payoff is the number of moves, say, of P until capture. An equiprobable distribution is probably the most desirable way to get the play started.

12.5. THE IMPORTANCE OF APPROXIMATIONS

Almost all the investigations on game theory up to the present have been concerned with precise statements, but there are essential domains in the subject where the exact solution has but a negligible practical advantage over an approximation thereof. Of the two chief such categories we discuss below, the first pertains not only to games but to the whole general field of maximization.[13] The second applies to games of incomplete information whose solution entails a mixed strategy.

1. The principle of flat laxity

Although we have ventured to affix a name to this principle, it is so obvious that there is embarrassment even in its mention. It lies under the nose of every student of the elementary calculus, yet we can recall no instance of its explicit delineation.

The student learns early to maximize a function by seeking points where its derivative is zero. The obvious principle we wish to asseverate is merely this:

At such maxima the value of the argument of the function is generally not critical. This is simply because the derivative—interpreted in the elementary way as a rate of change—is zero. Elementary as such ideas may be, three types of maxima are shown at (a) of Figure 12.5.1: one with zero

[13] As is well known, logically equivalent to minimization, for we need but maximize the negative of a quantity to minimize it.

derivative and two other common types, the third being at the endpoint of the domain of the argument. Observe the relative changes in function value for the same deviation of the argument from its value at the maximizing point in the three cases.

Solutions in the variational calculus which are integrals of the Euler equation likewise often belong to the zero derivative genre; we shall call the whole class of this type of maxima (and minima) *flat*. They are all subject to the title principal: small deviations of the argument from its optimizing value are uncritical.

It is highly likely that the flat laxity principle accounts for the astonishing dearth of practical applications of the many elegant mathematical maxima that have appeared throughout scientific history.[14]

This discussion can be applied to saddlepoints and so to game theory. In differential games, the principle supplies some pragmatic advice. In those cases with interior maxima or minima of the control variables, the gains from a too scrupulous adherence to an optimal strategy may not be

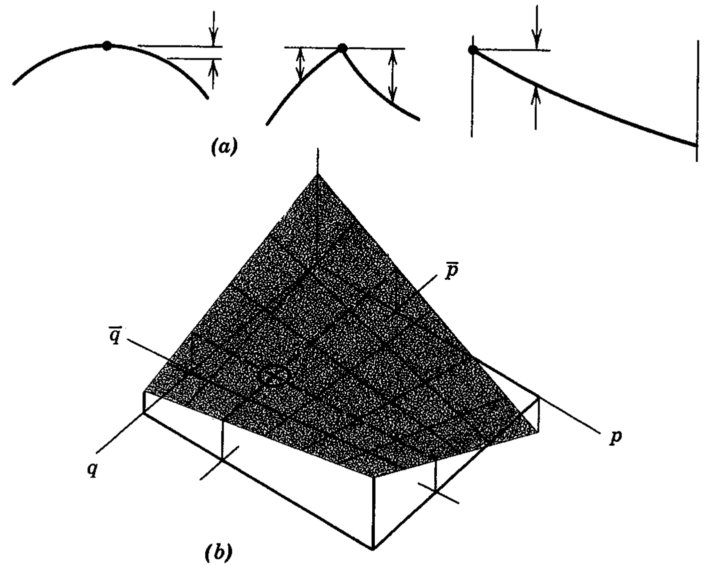


Figure 12.5.1

[14] Examples: The right circular cylinder of a given volume has minimal surface area when the height is equal to the diameter. However, tin cans of this proportion are seldom seen on market shelves.

Munk's elliptical wing planform of minimal induced drag seldom appears in aircraft design.

worth while. But in those cases with extreme optimizing values, such as those with linear vectograms, the gains from optimal play can be very genuine.

Finally, let us look at games of incomplete information with mixed optimal strategies. For simplicity, we think of a case with a finite, discrete matrix. A typical mixed strategy solution demands that one player play a certain *subset*[15] of n of his strategies with *positive* probabilities $p_1, \ldots, p_n$ ($\Sigma p_i = 1$), his opponent doing the same with $q_1, \ldots, q_m$. The positivity of the p_i and q_j implies that each is an internal optimizer and therefore of the flat type. The principle here grants tolerance for small errors. For example, a plot of the payoff, when $m = n = 2$, as a function of the mixing probabilities p and q (we use p, $1 - p$, instead of p_1, p_2, etc.) might appear typically as in Figure 12.5.1b; note that at the saddlepoint the surface has a flat tangent plane.

But matters can be otherwise for strategies not in the foregoing subsets. For such the $\bar{p}_j$ and $\bar{q}_j$ (see the footnote) are zero and so are extreme optimizers with possible sensitive responses to change. This is a commonplace rather than a pedantic matter, for a complete game matrix includes all strategies, even the absurdly bad ones. The latter are rigidly eschewed by even mediocre human contestants. A bridge player does not waste his trumps, not even with a small positive probability.

2. Probablistic indefiniteness

Again we suppose a game of imperfect information with mixed optimal strategies (or at least for one player). The actual usage of such strategies makes each partie a game of chance. Changing the strategies shifts the odds.

The second argument for the adequacy of approximate solutions is a facet of an old and basic question in the realm of probability. What is the actual effect of the probability distribution over a set of alternate outcomes when the number of trials is small?

If a gambler is betting repeatedly on a certain event and if the probability of his winning is actually .45, although he thinks it is .50, it will take him a great many bets to detect his error. The science of statistics is largely concerned with inferences drawn from such repeated trials and the subject is not simple. We must not attach unwarranted importance to the law of large numbers, which here asserts that for a *very long* series of bets the gambler is almost certain to win very close to 45% of the time.

[15] We wish to stress the fact that this is very often a proper subset; certain strategies not being used at all. Such cases can, and usually are, subsumed under generality by assigning zero probability to the latter strategies, but we have grounds here for distinguishing them.

The gambler might be a player in our subject game[16] and the shift in his win probability due to his opponent's improving his mixed strategy. For a small number of plays, then, what value a close calculation of an optimal strategy?

For games with mixed optimal strategies, both principles **1** and **2** can be effective and so the two effects are compounded. In certain extreme cases the composite might even make the range of approximation so broad that game theory has not much utilitarian to offer. The practical analyst should have a way of recognizing such cases.

Let us take as an example a moving target, say an aircraft or ship, being fired on by the enemy. As a game, the payoff is to express whether or not he scores a hit. The target has certain maneuvers at his disposal making his position hard to detect or predict. This choice constitutes his strategy, which in this game, whose essence is information, will certainly be mixed. The expected payoff will be the hit probability.

Suppose that by an improvement in his strategy the target can reduce the hit probability by, let us say, 5, 10, 25, or 50%. For an isolated, or few instances, the situation is like that of the gambler. Will a reduction of the odds by 5 or 10% matter very much? For 50%, yes. For 25%?

But there may be repetition. During a war there may be numerous engagements of targets and weapons; the hit probabilities approach the fraction of targets destroyed. Is it worth while to reduce the percentage of aircraft or ships lost by 5%? By 10, 25, or 50%?

Such are our grounds for the conclusion that the practical goal of game theory should be, in certain cases, especially those with sparse information, approximations. There seems to be little *proof* in support. In fact, the search game with many immobile hiders in Section 12.3 is the only instance we know. In Section 12.4, we conjectured that, with mobile hiders, all the steering aspects of the strategies are largely irrelevant. A proof of this and similar results, where they are true, would mark a decided and useful advance in game theory. Thus one essential problem is somehow to assess the degree of approximation appropriate to various classes of games.

But even when we feel satisfied that an approximation will be the thing for a certain game, how do we find it? We must be wary of the techniques of such subjects as mathematical physics. For we are dealing with conflicts, and there is always the opponent ready to exploit to the uttermost any deviation from the optimal. As the whole theory is predicated on his always acting rationally, we must posit that he will do so.

If one player, for example, is defending a city with a hundred gates and his strategy consists of an allocation of his forces amongst them, he will

[16] Which must be a win-or-lose type (two-valued payoff) for a perfect paradigm, but the idea is clearly very general.

have a close approximation to an excellent strategy if he adequately guards ninety-nine. But if his opponent has good intelligence, all his forces will surge through the neglected gate and the near perfect defense is useless.

So the techniques of approximation will require some innovation. They must be proof against the worst advantage the opponent can take of them.

Although we would encourage the investigation of approximations, we do not advocate no other approach. If exact optimal mixed strategies were found for, say, the search games of Sections 12.3 and 12.4, although their actual use may gain a contestant little, their discovery would probably throw a floodlight on an area of dark ignorance.

12.6. THE CHANCIFYING METHOD

The name with which we have dubbed this technique of solving certain games has a clumsy ring. But the same adjective perhaps fits the method itself; however, for a certain class of models we know of no other.

Suppose a zero-sum two-player game of incomplete information has "steady state" or "stationary" character. This concept is defined most readily if we adopt a discrete model. The game is *stationary* if the decision pattern recurs cyclically and at each cycle either the partie terminates or the situation, aside from the choices made, is the same as during earlier cycles. That is, a spectator who began witnessing a partie in midplay could not tell how long it had been going on.

Let us say that P, the minimizing player, moves first. If necessary, we must attach an artificial "past" so that the opening position is indistinguishable from one in midplay. We alter the rules so that instead of P making the first decision, it is done in accordance with stipulated probabilities $x_1, \ldots, x_n$, ($\sum_i x_i = 1$) indicated collectively by x. That is, the first move is now a chance one. The Value of this new game then depends on x and will be denoted by $\phi(x)$.

By applying the rules to the first full cycle of moves, it is often possible to obtain a functional equation that ϕ must satisfy. If its solution is essentially unique so that ϕ can be ascertained, the Value of the original game will be

$$V = \min_x \phi(x).^{17}$$

For the sole distinction in the original game is that P has the choice of x and he will exercise it to minimize V.[18]

[17] Sometimes inf $\phi(x)$, for there are cases when a player has no optimal strategy, that is, he cannot attain the Value V but has strategies that guarantee payoff arbitrarily close to V.

[18] For a further instance see Reference [14].

Example 12.6.1. A simple aiming and evading game. This game is a discrete model of the type of Example 12.2.3, in fact, the simplest possible such that is not trivial. In Section 12.7 we shall briefly return to these games.

A counter rests on one of a doubly infinite row of points. At each of his moves P has the choice of moving it right or left to the adjacent point. The moves alternate with those of E, who at each turn has the choice of *waiting*—doing nothing—or "*firing*." If E elects the latter choice, he selects a point. The play then terminates; if the counter is on the point guessed E wins; if not, P does.

Basic is the incomplete information structure—the time lag. Just prior to his turn E knows all moves of P *except the two most recent.* Thus when he fires E knows that either the counter is at the last observed point (P moved RL or LR) or it is two points to the right [left] (P moved RR [LL]). And if rational, he will fire only at one of these three points.

When P and E play mixed strategies, which we may assume, the payoff becomes the probability that E scores a hit, that is, guesses correctly the location of the counter.

Let us chancify on P. Suppose he normally has the opening move. We imagine that P has had a preceding move, say from the left. We replace P's move by a chance one which is to the left with probability x and right with $1 - x$ ($0 \leqslant x \leqslant 1$). The Value, the minimax of the hit probability, we will call $\phi(x)$, which satisfies

$$\phi(x) = \min_{c,d} \max \begin{cases} xc \\ x(1-c) + (1-x)(1-d) \\ (1-x)\, d \\ x\phi(c) + (1-x)\phi(d) \end{cases} \tag{12.6.1}$$

Here the max applies to the four lines on the right, while min ranges over c, d ($0 \leqslant c, d \leqslant 1$).

To establish (12.6.1), at least heuristically, first, let P make the move following the chancified one as follows: If the counter was moved to left (probability $= x$), he next moves left again with probability c and right with $1 - c$. Similarly, if the chancified move was to the right, he continues right with probability d. The quantities c and d are to be part of P's optimal strategy and will be fixed later.

The four lines after the brace correspond to E's four possible responses. If he fires at the leftmost point—two spaces left of the last observed position—he scores only if the counter is there, such requiring two left moves, and the probability of this and so of a hit is xc. The next line is similarly the probability of the counter's being at the central point, the

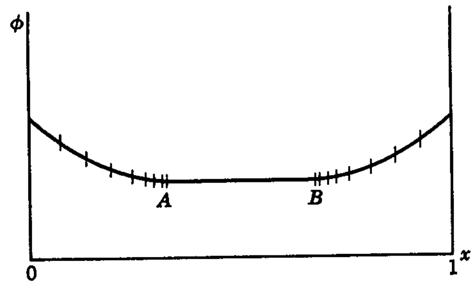


Figure 12.6.1

preceding moves having been LR or RL. Thus the hit probability, if E fires at the central possibility, is the second line in the brace. The third line corresponds to E's rightmost firing.

Finally, the last line is the payoff if E waits. Had the counter moved left, with probability x, we are faced with a recurrence of the original situation: the start of a chancified game with c replacing x. Its Value is thus $\phi(c)$. As with probability $1 - x$, E likewise confronts a game with Value $\phi(d)$; the fourth line is his expectation of a hit if he waits.

Now for any given c and d, E will select the maximum of the four entries. Then optimal play will dictate that P choose c and d as to minimize this max. Such will yield the Value, which is $\phi(x)$.

The solution $\phi(x)$ to (12.6.1) turns out involved. In a central segment AB of its graph (Figure 12.6.1) ϕ is constant, but in both extreme sections the graph consists of infinitely many straight segments whose endpoints have limits at A and B.

We have solved a few other similar problems and resulting ϕ was similarly intricate. It's ascertainment was tedious. And all we need of it to solve the original game is its minimum!

But is there not a hint here of a possible technique of approximation? The polygonal portions of the graph in Figure 12.6.1 strike the eye as being close to smooth curves. Is there an approximate method such that the counterpart of ϕ which it embodies is some simpler function?

The above game is one for which the arguments for approximation **1** and **2** of Section 12.5 both hold. Hence, for both it and various more realistic versions of the same genre, some of which are of considerable practical importance, a well-reasoned and sensible approximate solution would suffice and be of great value.

Further details on this game appear in References [13].

Appendix

Except for the first section, the appendix consists of a diverse selection of further examples of differential games. The solutions are sketched with hints for aid when unusual features or formal difficulties occur.

A1. A HIT PROBABILITY PAYOFF

In certain military games, such as those in Sections A2 and A5, one player, say E, continuously fires a weapon at his opponent. The payoff is to be the total probability of a hit.

The hit probability density p is supposed given: p is a function of the state variables (and conceivably of the control variables also). Thus in a definite partie, p becomes a function $p(t)$ of the time, and the probability of a hit in the interim $(t, t + h)$ is $hp(t) + O(h^2)$.

LEMMA A1. In the above circumstances the game has an integral payoff, and we may take

$$G = p. \tag{A1.1}$$

Proof. Let $Q(t)$ be the probability that the target is missed by all fire during $(0, t)$.

The probability that the target is missed during $(0, t + h)$ is the product of the probabilities that it was missed during $(0, t)$ and $(t, t + h)$. Thus if h is small,

$$Q(t + h) = Q(t)(1 - hp(t))$$

or

$$\frac{Q(t + h) - Q(t)}{h} = -p(t)Q(t)$$

which in the limit is

$$Q' = -pQ.$$

As $Q(0) = 1$, we have

$$Q(t) = \exp\left(-\int_0^t p(u)\,du\right).$$

Thus the probability of a hit during $(0, t)$ is $1 - Q(t)$ and so is an increasing function of the integral. Then if a strategy renders the integral, say, maximal, it likewise so renders the hit probability.

Note that the formal Value $V(\mathbf{x})$ yielded by the solution is not the hit probability, the actual Value. The latter is $1 - \exp(V(\mathbf{x}))$.

Let us think of the continuous fire as (it actually is) a succession of small shots. It is not hard to show that in two-dimensional space, the hit probability of a single shot varies inversely with the range, the distance from weapon to target, for reasonably large r. Thus p may be taken as a/r for some constant a. In three-dimensional space, correspondingly, $p = a/r^2$. (The former is used in the two succeeding problems only because it facilitates the formal mathematics.)

A2. THE FIXED BATTERY PURSUIT GAME

Both players have simple motion in a plane. The pursuer P is faster and could capture E readily were he not deterred by a barrage of gunfire assisting E. The firing is continuous and emanates from a battery located at a point O. The instantaneous hit probability is inversely proportional to the distance OP, and its time integral, as in Section A1, is to be the payoff.

Thus P seeks to capture with a minimum chance of being hit; he must choose a course which compromises between a direct chase of E and a wariness of being too close to the battery for too long. Similarly, E must inculcate into his flight from P maneuvering which lures P into lethal proximity to O.

We shall use the polar coordinates r, θ for P and the rectangular coordinates x, y for E, both systems having the same origin O with the lines $\theta = 0$ and $y = 0$ coinciding. The speeds of P and E shall be 1 and w with $w < 1$, so that w is really the speed ratio of E to P. The control variables ϕ and ψ are as in Figure A2.1. The KE are thus

$$\dot{r} = \cos\phi$$
$$\dot{\theta} = 1/r \sin\phi$$
$$\dot{x} = w\cos\psi$$
$$\dot{y} = w\sin\psi.$$

The payoff is integral with

$$G = \frac{a}{r}$$

a being a fixed positive constant. The terminal surface is the set where $|PE| = l$, a given positive number, and $\mathscr{E}$, the set with $|PE| \geqslant l$.

The solution is obtained almost entirely by pure integration, this example being very exceptional in its freedom from singular phenomena. One singular surface there certainly is: when P, O, E lie on a straight line in this order, the two symmetrical ways that P may skirt O clearly imply a dispersal surface, with an instantaneous mixed strategy.

Our standard method then presents no difficulties (except possibly the common one of deciding the sign of the V_i of the initial conditions). From the RPE, it will follow at once that E always travels a straight path. Any route of P, however, lies on a curve:

$$\frac{1}{r} = c_1 \sinh(\pm\sqrt{K}(\theta - c_3)) + c_2 \cosh(\pm\sqrt{K}(\theta - c_3)) + c_4$$

where $c_1, \ldots, c_4$, K are constants; K may be of either sign, meaning that for some paths sin, cos, replace sinh, cosh.

Figure A2.2 is a scale plot of typical partie. Here $w = \frac{2}{3}$; the dots indicate corresponding values of τ. Naturally any corresponding pair may be considered as a starting position.

Exercise A2.1. Write the KE for this problem using three state variables instead of four.

Exercise A2.2. Solve a version of the preceding problem which differs from it in that the battery, instead of being at a point, is distributed

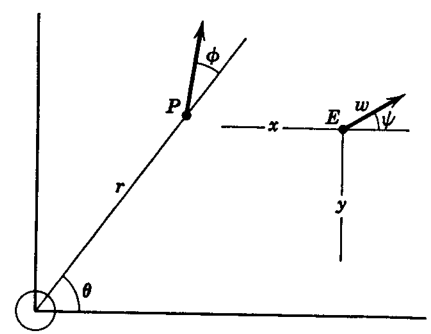


Figure A.2.1

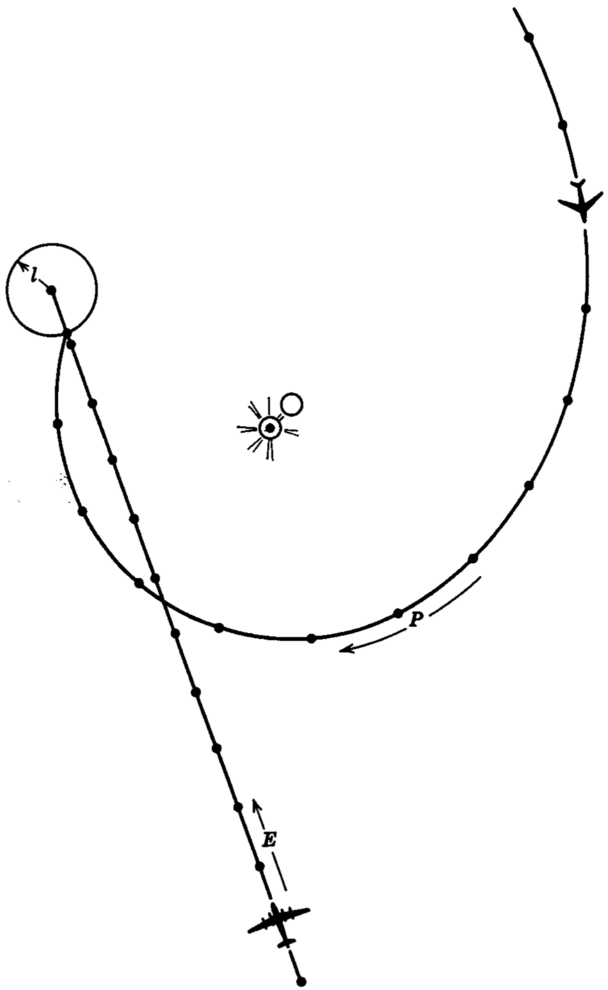


Figure A2.2

uniformly over a line $\mathscr{L}$. That is, $G = a/d$, where $d =$ distance P to $\mathscr{L}$.

Research Problem A2.1. Suppose in the problem of the text we had taken for G a function more rapidly decreasing than the inverse of r. Intuitively it might appear that, were this decrease rapid enough and had E time enough, the best strategy for E would be to get near O and to loiter there. Is this ever actually the case? If so, how is such a problem solved?

A3. OPTIMAL TRAJECTORIES OF GUIDED MISSILES

The design of long-range missile trajectories so as to maximize efficiency is a task amenable to the present methods. We are faced with a one-player game, the payoff or quantity to be optimized usually being the fuel consumption.

Let us look at a prototype case. The coordinates of $\mathscr{E}$, the state variables, will be such quantities as

the positional coordinates of the missile
the velocity components
the yaw angle and other such inclinations
the current weight of the missile (decreasing due to fuel consumption).

The control variables will be, of course, just those quantities regulated by the guidance system.

Let us say we are dealing with an ICBM designed to go from one point on the earth's surface to another distant one. It will have two or more stages of powered flight, possibly jettisoning some of its bulk at the end of each and concluding with a free fall stage.

In our approach to the problem, we work as usual retrogressively. Starting from the given destination, we ascertain the set of state variables enjoying the property that free fall from any such condition will bring the missile to target. The set will comprise a surface $\mathscr{C}_1$ in $\mathscr{E}$. Using $\mathscr{C}_1$ as a seat of initial conditions, we construct the retrograde optimal paths by our usual methods; the result solves the last powered stage. This solution is extended back until it fulfills the condition of the preceding stage juncture. Such states form a new surface $\mathscr{C}_2$, with the Value as a function on it. Similarly, we proceed until we reach the stage containing launch.

Unlike many of the other problems we have treated, this one has enjoyed a great deal of attention during recent years. Many analysts have obtained first rate results without having heard of differential games. Can we claim any advantages for the approach just sketched?

Within the pure logic of method, no.

For one always begins by setting up a model, which is a more or less simplified interpretation of reality. It is to the model and to it alone that the mathematics is applied. Generally, and almost certainly here, the underlying problem will be set so as to have a unique answer. Then all techniques which obtain this one answer are equally valid. Indeed, there should exist means of rendering them logically equivalent.

But in regard to the following two questions, differential games has advantages over at least some alternate approaches. How much light does the procedure shed on aspects of the general situation other than the mere trajectory? Does the underlying mathematics suggest a treatment of a more realistic model of greater intricacy?

First, through the concepts $\mathscr{C}_i$, as discussed above, the missile is not proscribed to a prescribed path. The vehicle is routed optimally to the optimal point of transition to free fall. (Of course, this is also achieved through classical calculus of variations with variable end conditions. Thus our point seems a minor one: that this consideration is more indigenous to differential games.)

The second advantage of the differential games approach is that the missile is not rigorously confined to a single navigational program. Should some untoward event put it off course, our craft does not strive to regain its old trajectory but assumes the optimal flight procedure pertinent to its new conditions. Mathematically this situation occurs because our method leads to the calculation of V and hence of the control variables (expressible in terms of the partial derivatives of V) throughout $\mathscr{E}$. This means that whatever conditions (state variables) should happen to arise during flight, we know the best way to navigate *at these conditions.*

Example A3.1. A simplified ICBM trajectory. The simplifications are fairly drastic, but nothing of principle is sacrificed. Our model is illustrative rather than realistic. We assume

A single stage of powered flight, followed by one of free fall.
A flat earth; the gravitational force is uniform and vertical. No air friction.
The missile thrust is of constant magnitude; steering is achieved by varying the direction of the thrust vector.
The realistic space is two dimensional; the missile always remains in the vertical plane containing the launching site and target.
The loss in weight of the missile due to fuel consumption can be neglected.

Thus in the xy-plane this missile is to be launched from the origin O with zero velocity. The target, labeled R in Figure A3.1, is at $(R, 0)$. The missile proceeds under powered flight (solid curve) to the point K where free fall begins.

The payoff to be minimized is the energy consumed. Due to our assumptions of constant thrust and missile weight, this is tantamount to minimizing the duration of powered flight.

The state variables will be x, y, the missile position coordinates, and u, v, its velocity components (see Figure A3.1). The thrust per unit mass will

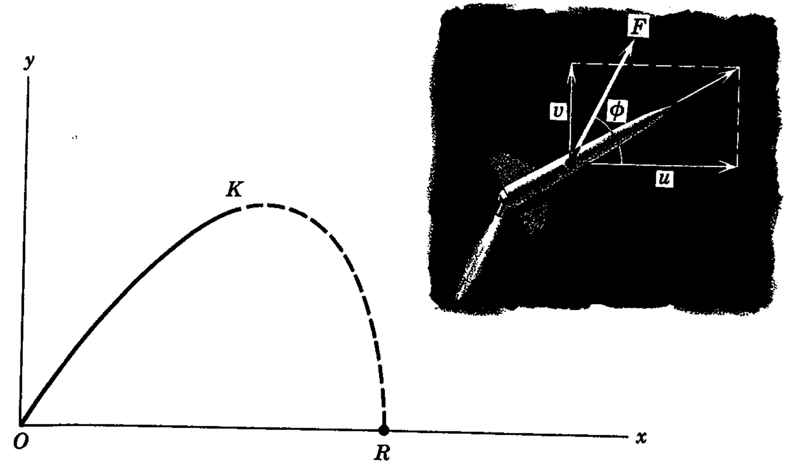


Figure A3.1

be a vector of length F and inclination ϕ, the latter being the single control variable. The KE are in this case the common Newtonian equations of motion (g = the usual gravity constant):

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{u} = F\cos\phi$$
$$\dot{v} = F\sin\phi - g.$$

We will construct $\mathscr{C}$ (the $\mathscr{C}_1$ of the preceding discussion), which is the set of all possible points of $\mathscr{E}$ where free fall can begin; K of the figure represents the x, y components of a typical one.

A little elementary dynamics leads to the representation:

$$x = R - s_1 s_3$$
$$y = -s_2 s_3 + \tfrac{1}{2} g s_3^2$$
$$u = s_1$$
$$v = s_2, \qquad s_3 \geqslant 0,\ s_1 \geqslant 0.$$

The reader can easily verify that if a body starts a free fall from the above position (x, y) and with initial velocity (s_1, s_2), at the time s_3 it will land at the point $(R, 0)$.

Forming the ME and RPE, then integrating the latter with the above initial conditions, leads to the paths:

$$\begin{aligned} u &= s_1 + F^2 s_3 \lambda(-s_2 + g s_3)\tau \\ v &= s_2 + (F^2 s_3 \lambda s_1 + g)\tau \\ x &= R - s_1 s_3 - s_1\tau - F^2 s_3 \lambda(-s_2 + g s_3)\tfrac{1}{2}\tau^2 \\ y &= -s_2 s_3 + \tfrac{1}{2} g s_3^2 - s_2\tau - (F^2 s_3 \lambda s_1 + g)\tfrac{1}{2}\tau^2 \end{aligned} \tag{A3.1}$$

where

$$\frac{1}{\lambda} = F s_3 \sqrt{s_1^2 + (s_2 - g s_3)^2}.$$

Also, $V = \tau$ and for the optimal strategy

$$\tan \bar{\phi} = \frac{s_1}{g s_3 - s_2}.$$

Thus the thrust direction will be constant throughout the powered flight, a point to which we shall return later. The trajectory will therefore be a parabola with axis parallel to the resultant vector of the thrust and gravity.

To proceed further involves the process of solving the system (A3.1) for s_1, s_2, s_3, and τ. The elimination leads to a fourth degree equation, and all the sought quantities are simply expressible in terms of the proper one of its roots.

But if we are concerned with the particular trajectory of Figure A3.1, for which the starting conditions are

$$x = y = u = v = 0$$

we are led to the actual flight trajectory easily. We can easily obtain a linear approximation of V in the neighborhood (in $\mathscr{E}$) of this trajectory, for the partial derivatives of V will figure in the complete integrals of the RPE.

Carrying through this special elimination leads ultimately to

$$S^3 - \frac{F}{g}(2S^2 - 1) = 0$$

where $S = \sin \bar{\phi}$. Thus the thrust angle is a function of F/g. It is easy to plot this relation by inverting a plot of F/g as a function of S. Such is done in Figure A3.2.

The quantity F/g can be regarded as the magnitude of the thrust when the weight of the missile is taken as the unit of force. It must exceed 1 in order for the craft to leave the ground. When it is very large, the plot shows that $\bar{\phi}$ is near 45°, a conclusion to be expected, for this is the well-known ballistic firing angle of maximal range (in a vacuum) and very large thrusts are of very short duration. As F/g diminishes toward 1, the thrust angle increases, approaching vertical launch.

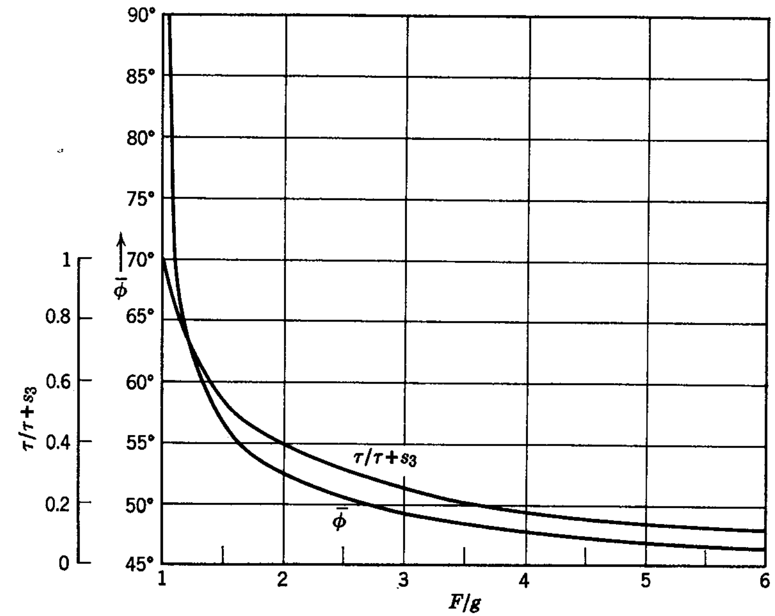


Figure A3.2

On the same figure we have plotted $\tau/(\tau + s_3)$ [$= S/(F/g)$]. As this quantity is the ratio of powered flight to total flight time, it serves a poignant descriptor of the total optimal trajectory. With large thrusts, the duration of powered flight is small; almost all the route is free fall. The two times are equal when F/g is about 1.6, and for still smaller thrust the powered phase predominates.

We saw that in this problem travel in least time implied a constant thrust direction. But such is not always the case, even with our simple (or no) gravity force field, no friction, etc. From a general integration of the RPE, we obtain

$$\tan \bar{\phi} = \frac{V_v}{V_u} = \frac{C_1\tau + C_2}{C_3\tau + C_4} \tag{A3.2}$$

where the C_i are constants.

If the object of the craft is simply to reach a given *location* in minimal time, then $\bar{\phi}$ will be constant. But the general $\mathscr{C}$ that we have been considering entail u, v as well as x, y, that is, terminal velocity as well as position count. It is clear that the craft when approaching $\mathscr{C}$ must anticipate its ultimate attitude before arrival. The form of (A3.2) is plausible in this regard. For when τ is large ($\mathbf{x}$ far from $\mathscr{C}$), $\bar{\phi}$ will be nearly constant, but as **x** nears $\mathscr{C}$, the thrust vector shifts direction in anticipation of its terminal conditions.

We stress this point to indicate that the simple anwer to our problem was perhaps more coincidental than indigenous. It involved a constant $\bar{\phi}$ even though velocities were germane to $\mathscr{C}$. However, had the problem been altered in any of a number of simple ways—air friction in the free fall, multistage flight, etc.—it is likely that (A3.2) rather than constant $\bar{\phi}$ would optimize.

Problem A3.1. For a projectile steered by a variable direction thrust, as in the preceding example, in a constant force field, show that a necessary and sufficient condition that the least time paths to $\mathscr{C}$ entail constant $\bar{\phi}$ is that on $\mathscr{C}$

$$V_x V_v - V_y V_u = 0. \tag{A3.3}$$

In particular, show that this condition holds if the terminal conditions stipulate location but not final velocities.

Research Problem A3.1. If instead of the constant force field, there is one derivable from an arbitrary potential $G(x, y)$,[1] what are the counterparts of the navigational schemes $\bar{\phi}$ = constant and (A3.2)? A case of particular interest would be the inverse square field.

Research Problem A3.2. What is the most economical way to send up a satellite to *any* circular orbit about the earth? We have in mind here an academic but interesting model. Make the same flight assumptions as earlier in this section except that gravity is radial and to vary inversely as the square of the distance from the center O of the earth. For $\mathscr{C}$ we use the set of all conditions at which the missile will be in a circular orbit about O (centrifugal must balance gravitational force).

Supposing the earth perfectly penetrable, under what conditions will the orbital radius be greater than that of the launch point (assumed on the surface of the earth) so that the orbit will be a possible one? Even with possibility in this sense, the trajectory may still penetrate and then leave the earth. If so, it might mean that satellites should be launched from mountain tops.

A4. AN ILLUSTRATION FROM CONTROL THEORY

We have noted that the modern theory of control seems to be subsumed in the topic of one-player differential games. The following simple yet typical instance is borrowed from Hale and Lasalle.[2]

[1] So that two of KE become $u = F\cos\bar{\phi} - G_x$, $v = F\sin\bar{\phi} - G_y$.

[2] Differential Equations: Linearity vs. Nonlinearity, *SIAM Review*, Vol. 5, No. 3 (July, 1963).

A body of unit mass moves lineally in a fliud of unit viscosity (the viscous drag is the negative of the velocity). We may exert control by applying a force to the body whose magnitude may not exceed 1. The object is to bring the body to rest at a prescribed point O in the least possible time.

Let x be the positional coordinate, of the body, with O at $x = 0$, and let y be the velocity. The kinematic equations then are

$$\dot{x} = y$$
$$\dot{y} = -y + \phi, \qquad -1 \leqslant \phi \leqslant 1.$$

(The second states: acceleration = viscous force + control force.) The payoff is integral with $G = 1$.

The authors' solution is shown in Figure A4.1.[3] The two curves C are semiuniversal; they are the only paths reaching the origin; all others are tributary to one of them.

The genesis of the semiuniversal curves is interesting. Our method requires that we take as $\mathscr{C}$ a circle of radius l centered at $x = 0$, $y = 0$ and

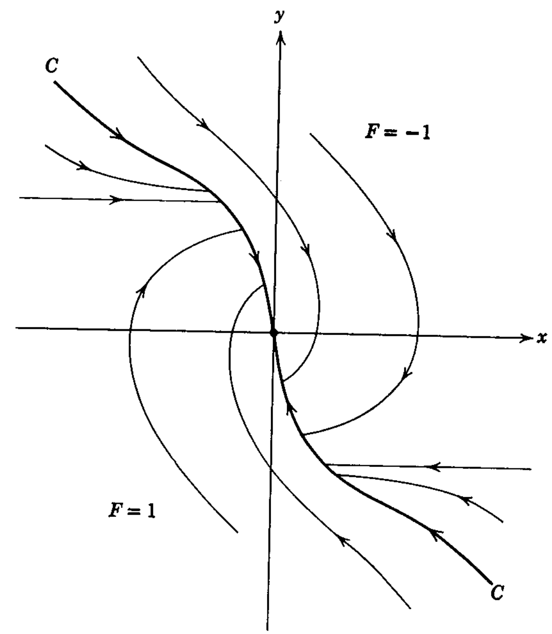


Figure A4.1

[3] Their F equals our ϕ.

later let $l \to 0$. With positive l there will be a full family of paths from $\mathscr{C}$ with two transition curves. As l becomes 0, the paths coalesce into just the two shown. The transition surfaces must coincide with them and so become semiuniversal.

Research Problem A4.1. The same as above except that now we wish to minimize the energy needed to arrest the body at O. As such is $\int \phi\, dx$, we take $G = \phi y$, the only formal change.

A5. THE BOMBER AND BATTERY GAME

This contest illustrates the astonishing variegation that may be hidden in apparently simple military (or other) games. The full solution splits into nine separate cases, and even some of these admit of further logical subdivisions. We shall but indicate this panorama here with emphasis on some especially instructive points.

An attacking craft, steered by P, moves in the plane with simple motion at speed v. His destination is E's territory, which is bounded by a coastline, a curve $\mathscr{L}$. From a battery fixed at a point O, E fires at the invading P, in accordance with Section A1, so that the payoff is the hit probability.

Each player will be subject to a distinct type of constraint.

For P it will be the flight time, fuel, or path length, all equivalent because of his constant speed. We select the first: total flight time $\leqslant T$. This constraint is of the type discussed in Section 5.7.

For E we shall limit the amount of ammunition m he has available for firing at his target P. Let c be the maximal rate at which E can fire. His control variable ψ will be the fraction of this rate he selects at each instant. Then we include in the KE

$$\dot m = c\psi, \qquad 0 \leqslant \psi \leqslant 1$$

and replace $G = a/r$ (see Section A1) in this case by

$$G = \frac{a\psi}{r}$$

so that a diminished rate of fire is reflected in a current proportionately diminished weapon effectiveness.

Thus P must pick a path of length not exceeding vT, with the flight time T, prescribed from his starting point to $\mathscr{L}$ so as to minimize the probability of his being hit en route. During this flight, E fires at him, allocating his ammunition, of fixed amount m, over the interlude so as to maximize the hit probability.

The solution will probably not be of great practical importance, for the differences in probability for slightly different routes will likely be nugatory

(see the discussion of practical assessment in Chapter 11). But the instructive level is high: the diversity of phenomena is typical of other, especially military, problems and there is also the illustration of two different type constraints effective simultaneously.

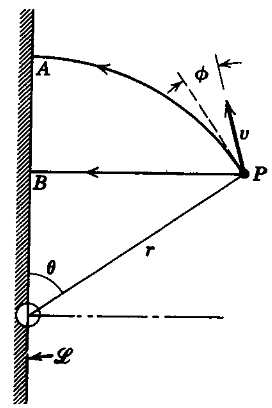


Figure A5.1

When not otherwise specified, the coast $\mathscr{L}$ will be taken as a straight line with the battery location O on it. For some purposes it will be more enlightening to place O forward of $\mathscr{L}$; we will then speak of advanced defense.[4] In the first instance below we take an arbitrary coastline.

The polar coordinates r, θ will describe P's location; they and his control variable ϕ are shown in Figure A5.1. The other state variables are T, the flight time allowed P, and m, the ammunition allowed E. Thus the KE are

$$\dot r = v \sin \phi$$
$$\dot\theta = -\frac{v}{r}\cos\phi$$
$$\dot m = -c\psi, \qquad 0 \leqslant \psi \leqslant 1$$
$$\dot T = -1.$$

Besides the obvious strictures, $m > 0$, $r > 0$ on $\mathscr{E}$ we demand further:

$$vT \geqslant r \sin\theta \tag{A5.1}$$

so that P can always reach $\mathscr{L}$ (at least via PB in the figure) and

$$0 \leqslant \theta \leqslant \frac{\pi}{2}$$

on grounds of symmetry. It is obvious that $\theta = 0$ will be a dispersal surface (but no instantaneous mixed strategy is ever required).

There are two natural possibilities for the terminal surface:

$\mathscr{C}_1$: P is on $\mathscr{L}$. Thus the partie is over. There may or may not be some excess m. As a parametrization we may take

$$r = s_1 > 0$$
$$\theta = 0$$
$$m = s_3 \geqslant 0$$
$$T = 0. \tag{A5.2}$$

[4] The case of "retarded" defense, where O lies inland, promises little further novelty or interest.

Further, on $\mathscr{C}_1$, $V = 0$, and, in accordance with Section 5.7, we take

$$V_T = -\lambda,\ \lambda \geqslant 0$$

as the third parameter.

$\mathscr{C}_2$: Whenever $m = 0$, as there can be no further changes in payoff, P's strategy from here on is immaterial (we suppose T big enough to permit his reaching $\mathscr{L}$, that is, (A5.1)). Thus the plane $m = 0$ is a natural terminal surface. Our parametrization:

$$\begin{aligned} r &= s_1 > 0 \\ \theta &= s_2, \qquad 0 \leqslant s_2 \leqslant \frac{\pi}{2} \\ m &= 0 \\ T &= s_4. \end{aligned} \tag{A5.3}$$

The next three examples are instructive but extreme cases in that they lie on the fringes of the totality of solutions.

Example A5.1. No constraint on E's firepower. We assume m so large that E can maintain full fire ($\psi = 1$) throughout the partie. Then we may ignore m as a state variable. We take initial conditions on $\mathscr{C}_1$ (with the third of (A5.2) suppressed) and $G = a/r$. The problem is thus a pristine illustration of the integral constraints discussed in Section 5.7 with, of course, $L = 1$.

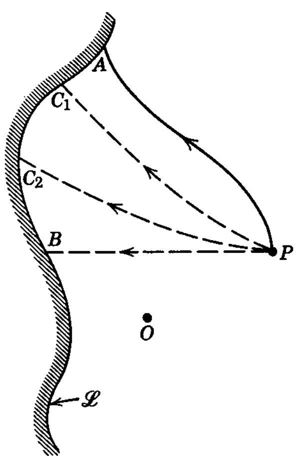


Figure A5.2

If we solve the system with $\lambda = 0$, we obtain a path, which, as we know, would be optimal for P even were he freed of the constraint on T. For general coasts, this path, such as PA in Figure A5.2, turns out to have an equation of the form $r = c_1 \exp(-c_2\theta)$ the c_i being constants.

For $\lambda > 0$ we obtain paths ($PC_1, PC_2, \ldots$ in the figure) where the constraint is effective, that is, the T allowed P is less than that needed to traverse PA. Finally comes the path PB when there is just enough T to permit travel on a horizontal straight line ((A5.1) holds as an equality).

In our standard case of a straight shore with O on it, the absolute optimal path PA (see Figure A5.1) reduces to a circular arc of constant r. But the

effectively constrained paths are given by

$$\begin{aligned} r &= \sqrt{X(\tau)} - \frac{a}{\lambda} \\ \theta &= v\left(\frac{a}{\lambda} + s_1\right)\int_0^{\tau} \frac{du}{\sqrt{X(u)}(\sqrt{X(u)} + a/\lambda)} \end{aligned} \tag{A5.4}$$

where $X(\tau) = (v\tau)^2 + (a/\lambda + s_1)^2$.

A closed expression for θ can be worked out from the general indefinite integral

$$\int \frac{dx}{\sqrt{X}(\sqrt{X} + \alpha)} = \frac{1}{2\sqrt{B^2 - C(A - \alpha^2)}} \log \frac{\Upsilon_1(Cx + B) + A + Bx + \alpha\sqrt{X}}{\Upsilon_2(Cx + B) + A + Bx + \alpha\sqrt{X}}$$

where $X = Cx^2 + 2Bx + A$ and the Υ_i are the roots of $X - \alpha^2$.

The following identity is a consequence of the RPE and ME and useful for integrating in almost all cases.

$$(r\overset{\circ}{V}_r) = -(c\bar{\psi}V_m + V_T) = \text{constant}. \tag{A5.5}$$

Example A5.2. No constraint on P's flight time. When P can fly a path of any length, but E is constrained by both a maximal firing rate c and limited amount m of ammunition, the optimal play is clear at once: P will travel directly away from O until the ammunition is exhausted; then he proceeds to $\mathscr{L}$ at his leisure. It is also clear that E commences firing at full rate ($\psi = 1$) at once and so perseveres until exhaustion.[5]

Exercise A5.1. Using $\mathscr{C}_2$, obtain this result analytically.

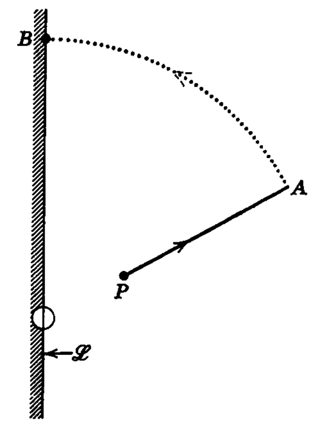


Figure A5.3

Of course, the above solution holds with a constraining T provided it is large enough to permit P, after his radial flight, to reach $\mathscr{L}$ without ever decreasing the distance OP. Precisely, we must have

$$T \geqslant \frac{m}{c} + \frac{\theta}{v}\left(r + \frac{vm}{c}\right). \tag{A5.6}$$

Equality limits P to the path PAB of Figure A5.3,[6] AB being a circular arc.

[5] If we were to envisage this problem removed from context, this solution seems ridiculous from a practical standpoint. Nevertheless it is mathematically correct. The unreality stems from the formulation of the problem, not its solution. Such formulations are a hazard for the tyro at military analysis and we urge the reader to reflect on this instance.

[6] Of course, A is part of the terminal conditions $\mathscr{C}_2$, and so the arc AB will not be part of the formal solution. Nevertheless it will make such an appearance a few pages hence.

Example A5.3. With limited ammunition and no maneuvering allowed the invader. If (A5.1) holds as an equality, P has no alternative to the straight route perpendicular to $\mathscr{L}$. The situation is then identical to that of Example 7.14.1 with its semiuniversal surface.

It is interesting to generalize to the advanced defense case in which O lies a distance D ahead of $\mathscr{L}$. First suppose m small. Then E should fire full blast ($\psi = 1$) while x ranges over an interval centered at O of appropriate length. But if $m > 2Dc/v$, $\dot{E}$ has ammunition enough to last while P traverses more than the distance $2D$. Then E should commence full fire at the latest possible time such that he can maintain it until P reaches $\mathscr{L}$. That is, he fires during the final interval of play of duration m/c. If $x \leqslant mv/c$, E opens fire at once; P will reach $\mathscr{L}$ with some m being surplus.

All this leads to the paths in $\mathscr{E}$, the (r, m)-plane, the only two state variables needed, shown in Figure A5.4, which the reader can readily interpret.

Research Problem A5. Is it possible to derive this manifestly correct solution *completely* by analysis?

We now investigate more basic solutions and seek a possible ψ—universal surface. The ideas of Chapter 7 require first a recasting into the terminal payoff form. The number of state variables is then five; our general technique is exceeded, but we can solve this game nonetheless.

The three equations (7.13.2) become in this case

$$cV_m + \frac{a}{r} V_u = 0 \quad (\alpha)$$

$$\min_\phi v\left[V_r \sin \phi - \frac{V_\theta}{r} \cos \phi\right] - V_T = -\rho v - V_T = 0 \quad (\beta)$$

$$\frac{V_u}{r^2} \sin \phi = 0 \quad (\gamma)$$

where u is the adjoined state variable.

Leaving aside the case of $V_u = 0$, obviously trivial, (γ) tells us that $\sin \phi = 0$. Such navigation implies a circular arc path about O with r constant.

If E had enough ammunition for perpetual full fire, we would be back to Example A5.1. Therefore we suppose that $\psi \neq 1$ always. We are faced with finding $\breve{\psi}$, the optimal way for E to distribute his limited fire power.

We assert $\breve{\psi}$ should be constant during the action and this constant rate should be such that E consumes all his m just as P reaches $\mathscr{L}$. For if E expended his ammunition nonuniformly, there would have to be at least one small interim when ψ was greater than the mean rate $\breve{\psi}$ and another in which ψ was less. If P were to take a short cut during the latter interim,

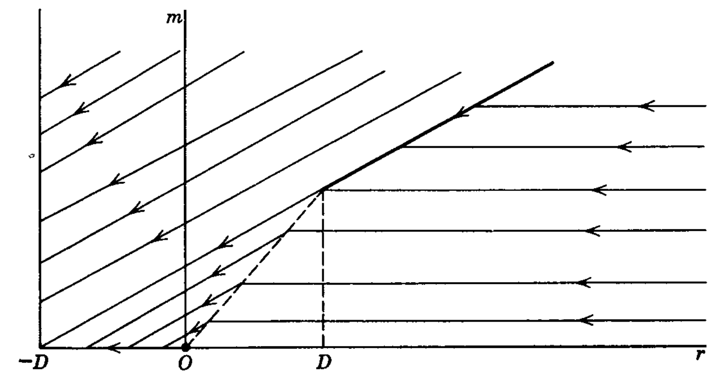


Figure A5.4

by flying the chord of a small arc of his optimal circular path instead of the arc itself, and, further, he uses the T saved thereby to increase r slightly during the former interim, then he betters (decreases) the payoff and E's strategy could not be optimal.

On the alleged universal surface r and $\breve{\psi}$ are constants, say s_1 and s_2. The parametrization can then be written (in the original $\mathscr{E}$):

$$\begin{aligned} r &= s_1 \\ \theta &= \frac{vs_3}{s_1} \\ m &= cs_2s_3 \\ T &= s_3 . \end{aligned} \qquad \text{(A5.5)}$$

On this surface, from the original definition of the payoff,

$$V = a\int_0^\tau \frac{\breve{\psi}}{r}\, dt = \frac{as_2s_3}{s_1}.$$

Problem A5.1. Show that the necessary conditions (α), (β), (γ) hold on the surface (A5.5). Prove by direct reasoning that it is a universal surface.

Using (A5.5) as initial conditions, we can integrate the RPE in the standard way with $\psi = 0$ and 1 to obtain the two sets of tributary paths.

The tributaries with full fire ($\bar{\psi} = 1$)

The integration shows that r is always an increasing function of τ, so that these paths meet the universal surface on its interior side. Note also that they are then distinct from those of Example A5.1 where r is nonincreasing in τ.

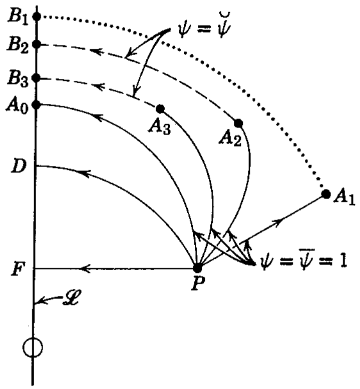


Figure A5.5

Not all the integrals will represent optimal paths. For those with $s_2 = 0$, which meet the universal surface at $\mathscr{L}$ with ammunition spent, would contradict Example A5.1 were they optimal.

We examine the limiting behaviour as $\breve{\psi} \to 0$. Among the path equations with $s_2 = 0$ we find

$$r = s_1 - v\tau, \qquad \theta = \frac{vs_3}{s_1}.$$

But such are the radial paths of Example A5.2. Further, we see that when the universal surface is reached ($\tau = 0$), then $m = 0$ and **x** continues on the circular arc with no firing ($\psi = 0$). Thus we have exactly the total path of Figure A5.3. This path is unique when E has no surplus T for any other. Thus this case marks the blending of this subsection with Example A5.2 as T increases through this critical value.

We now investigate what happens as T begins to fall below the above critical amount. The paths progress through the sequence PA_iB_i of Figure A5.5. On PA_i there is maximal fire ($\psi = 1$); on the circular arcs A_iB_i ($i \neq 1$), E fires at the constant rate $\breve{\psi} < 1$, which is such as to just exhaust the ammunition when P reaches $\mathscr{L}$.

Let us see what happens at the other limiting extreme as $\breve{\psi} = s_2 \to 1$. The tributary paths become circular arcs which lie in the universal surface. Such can only happen when

$$m = cT.$$

Thus there is no break between tributary and universal path.

To unify our ideas let us see what happens when P starts from a given starting point with a fixed but a sufficiently large m and T varies. The smallest allowable T admits only the direct path PF of Figure A5.5. Increasing T brings us continuously to the circular arc PD (as in Example A5.1) with a decreasing payoff. We suppose there is excess ammunition, that is, $m > 0$ at D. Greater T, as we know, leads to the PA_iB_i. As T is decreased in this set we must arrive at the path PA_0 with A_0 on $\mathscr{L}$ and $\bar{\psi} = 1$ throughout. This path, as we saw, is distinct from PD and is not optimal. Thus the payoff for PA_0 is greater than that for PD. But the payoff for $PA_1(B_1)$ is lower (it is the lowest possible). Therefore there must be an intermediate path, say PA_2B_2, for which V is the same as for PD.

Thus if T is increased over that for PD, for a certain interval PD remains optimal and V stays constant. This happens until T grows enough for P to traverse PA_2B_2; at this state P has two optimal strategies. (Is there a dispersal surface?) From then on increasing T leads through the PA_iB_i series with decreasing V until the ultimate at PA_1.

The greater m, the greater the gap. With infinite m, as Example A5.1 shows, PD is the best possible path and the gap has widened into non-existence.

Research Problem A5.2. Characterize the set of starting positions mentioned at which P enjoys two optimal strategies. Is the set attained by a construction as in Section 6.5 so that a dispersal surface results?

The tributaries with no fire ($\bar{\psi} = 0$)

Integration of the RPE shows that these paths are straight. They then must meet the universal surface tangentially and appear as in Figure A5.6.

The conditions for such are clearly

$$r\theta > vT$$

and $m <$ that required for continuous full fire over the dashed arc of the figure.

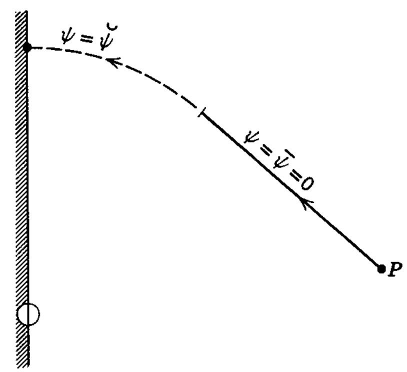


Figure A5.6

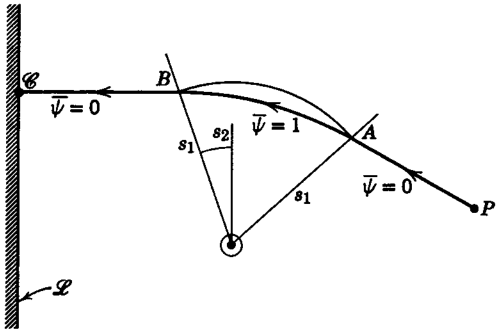


Figure A5.7

Observe the plausibility. The path is that which maximizes the minimal range occurring on it.

Finally, there must be still another class of solutions. Consider Example A5.3 where $vT = r \sin \theta$ so that P is confined to a straight horizontal path and m is not great, so that E fires fully in a final interval. We saw a resulting semiuniversal surface.

For a slight increase in T, the path neighbors the straight one and we should expect a qualitative similarity.

Such phenomena are more interesting in the case of advanced defense. The chief novelty here is that many optimal parties will end with m exhausted before P reaches $\mathscr{L}$ (the reason is intuitively obvious). A $\mathscr{C}_2$ is

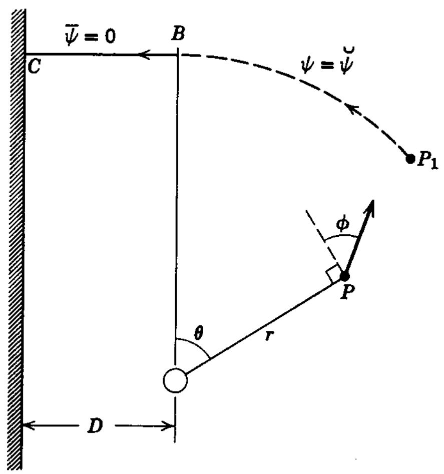


Figure A5.8

suggested which lies left of O and with T just enough to permit a horizontal flight to O.

Two classes of solutions result. One has paths on which r increases with t and which curve toward $\mathscr{L}$. The second entails a transition surface. A typical optimal partie appears in Figure A5.7. Here B is a point of the above $\mathscr{C}_2$ and A of the transition surface. The segments PA and BC are straight with no fire. The latter occurs only on AB, a curve which meets PA and BC smoothly, has (aside from A and B) $r < |OA| = |OB|$ and is symmetrical about its midpoint.

Although such paths resemble those of Example A5.3, we have not studied the details of the fusion with that case. They probably also merge with the $\bar{\psi} = 0$ tributaries of the universal surface.

The latter surfaces, when the defense is advanced, entail optimal play as in Figure A5.8. The arc P_1B lies on the universal surface, but it terminates at a set of points such as B. Left of here the optimal paths are, as above, horizontal with no fire.

The possible blending of the Figure A5.6 tributaries on such universal surfaces with the paths of Figure A5.1 is manifestly obvious.

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